

MTHSC 206 SECTION 15.1 – DOUBLE INTEGRALS OVER RECTANGLES

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IDEA

Suppose that we want to compute the volume underneath the graph of a positive valued function $f(x, y)$ as x and y vary over the rectangle $R = [a, b] \times [c, d]$.

Motivated by our success with functions of one variable, we subdivide R into small rectangles.

Set $\Delta x = \frac{b-a}{n}$ and $\Delta y = \frac{d-c}{m}$ for some integers m and n .

Let $x_i = a + i\Delta x$ and $y_j = c + j\Delta y$.

The volume underneath $f(x, y)$ and directly above the rectangle $R_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ can be approximated by

$$f(x_{ij}, y_{ij})\Delta x\Delta y = f(x_{ij}, y_{ij})\Delta A,$$

where (x_{ij}, y_{ij}) is a point lying in R_{ij} . Thus we approximate the volume underneath R by

$$V \approx \sum_{i=1}^n \sum_{j=1}^m f(x_{ij}, y_{ij})\Delta A.$$

FACT

If the function $f(x, y)$ is continuous then the volume lying below the graph of $f(x, y)$ and directly above the rectangle $R = [a, b] \times [c, d]$ is given by

$$V = \lim_{m, n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m f(x_{ij}, y_{ij}) \Delta A.$$

DEFINITION

We define the double integral of $f(x, y)$ over the rectangle $R = [a, b] \times [c, d]$ by

$$\iint_R f(x, y) \, dA = \lim_{m, n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m f(x_{ij}, y_{ij}) \Delta A$$

DEFINITION

A function f is called integrable if the limit in the previous definition exists.

NOTE

We could choose our point (x_{ij}, y_{ij}) to be the corner point (x_i, y_j) . Then the expression for the double integral becomes

$$\iint_R f(x, y) \, dA = \lim_{m, n \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m f(x_i, y_j) \Delta A$$

EXAMPLE

Estimate the volume underneath the graph of $f(x, y) = xy$ and directly above the rectangle $[1, 5] \times [1, 5]$. Can you compute the volume?

EXAMPLE

Let $R = [-1, 1] \times [-3, 3]$. Interpret the integral $\int \int_R \sqrt{1 - x^2} \, dA$ as a volume and calculate its value exactly.

MIDPOINT RULE FOR DOUBLE INTEGRALS

$$\int \int_R f(x, y) \, dA \approx \sum_{i=1}^n \sum_{j=1}^m f(\bar{x}_i, \bar{y}_j) \Delta A,$$

where $\bar{x}_i = x_{i-1} + \frac{\Delta x}{2}$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j = y_{j-1} + \frac{\Delta y}{2}$ is the midpoint of $[y_{j-1}, y_j]$.

EXAMPLE

Let $R = [0, 4] \times [0, 4]$. Use the midpoint rule to estimate $\int \int_R (x^2 + y^2) \, dA$.

NOTE

We can estimate the average value of the function $f(x, y)$ over the rectangle $R = [a, b] \times [c, d]$ by

$$\begin{aligned}\frac{1}{mn} \sum_{i=1}^n \sum_{j=1}^m f(x_i, y_j) &= \sum_{i=1}^n \sum_{j=1}^m f(x_i, y_j) \frac{\Delta A}{A} \\ &= \frac{1}{A} \sum_{i=1}^n \sum_{j=1}^m f(x_i, y_j) \Delta A.\end{aligned}$$

DEFINITION

We define the average value of the function $f(x, y)$ over the rectangle $R = [a, b] \times [c, d]$ by

$$f_{\text{ave}} = \frac{1}{A(R)} \int \int_R f(x, y) \, dA.$$

EXAMPLE

Estimate the average value of the function $f(x, y) = xy$ on the rectangle $R = [1, 5] \times [1, 5]$.

PROPERTIES OF DOUBLE INTEGRALS

- 1 $\int \int_R [f(x, y) + g(x, y)] \, dA = \int \int_R f(x, y) \, dA + \int \int_R g(x, y) \, dA.$
- 2 If $c \in \mathbb{R}$, $\int \int_R cf(x, y) \, dA = c \int \int_R f(x, y) \, dA.$
- 3 If $f(x, y) \geq g(x, y)$ for all points (x, y) in R then $\int \int_R f(x, y) \, dA \geq \int \int_R g(x, y) \, dA.$