MTHSC 206 Section 15.1 – Double integrals over rectangles

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The volume underneath $f(x, y)$ and directly above the rectangle $R_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ can be approximated by

$$f(x_{ij}, y_{ij})\Delta x \Delta y = f(x_{ij}, y_{ij})\Delta A,$$

where (x_{ij}, y_{ij}) is a point lying in R_{ij}. Thus we approximate the volume underneath R by

$$V \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}, y_{ij})\Delta A.$$

Fact

If the function $f(x, y)$ is continuous then the volume lying below the graph of $f(x, y)$ and directly above the rectangle $R = [a, b] \times [c, d]$ is given by

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Definition

We define the double integral of $f(x, y)$ over the rectangle $R = [a, b] \times [c, d]$ by

$$\int\int_R f(x, y)\, \mathrm{dA} = \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}, y_{ij})\Delta A$$

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Note

We could choose our point (x_{ij}, y_{ij}) to be the corner point (x_i, y_j). Then the expression for the double integral becomes

$$\int\int_R f(x,y)\ \mathrm{dA} = \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\Delta A$$

Example

Estimate the volume underneath the graph of $f(x, y) = xy$ and directly above the rectangle $[1, 5] \times [1, 5]$. Can you compute the volume?

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Let $R = [-1, 1] \times [-3, 3]$. Interpret the integral $\int\int_R \sqrt{1 - x^2}\, dA$ as a volume and calculate its value exactly.

Midpoint Rule for Double Integrals

$$\int\int_R f(x,y)\ \mathrm{dA} \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(\bar{x}_i, \bar{y}_j)\Delta A,$$

where $\bar{x}_i = x_{i-1} + \frac{\Delta x}{2}$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_i = y_{i-1} + \frac{\Delta y}{2}$ is the midpoint of $[y_{i-1}, y_i]$.

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Example

Let $R = [0, 4] \times [0, 4]$. Use the midpoint rule to estimate $\int\int_R (x^2 + y^2)\ \mathrm{dA}$.

Note

We can estimate the average value of the function $f(x, y)$ over the rectangle $R = [a, b] \times [c, d]$ by

$$\frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j) \quad =$$

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&= \frac{1}{A} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j) \Delta A.
\end{aligned}$$

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We define the average value of the function $f(x, y)$ over the rectangle $R = [a, b] \times [c, d]$ by

$$f_{\text{ave}} = \frac{1}{A(R)} \int\int_R f(x, y)\ dA.$$

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Example

Estimate the average value of the function $f(x, y) = xy$ on the rectangle $R = [1, 5] \times [1, 5]$.

Properties of Double Integrals

1. $\int\int_R [f(x,y)+g(x,y)]\ \mathrm{dA} = \int\int_R f(x,y)\ \mathrm{dA} + \int\int_R g(x,y)\ \mathrm{dA}.$

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3. If $f(x,y) \geq g(x,y)$ for all points (x,y) in R then $\int\int_R f(x,y)\ \mathrm{dA} \geq \int\int g(x,y)\ \mathrm{dA}$.