MTHSC 206 Section 15.10 – Change of Variables in Multiple Integrals

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Recall

In one variable calculus we recall the change of variable formula for integration is

$$\int_a^b f(x)\ \mathrm{dx} = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)\ \mathrm{du}$$

where we have substituted $x = g(u)$. We are assuming that g is one to one on $[a, b]$ and that g is continuous.

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&\approx \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)\,\mathrm{du}.
\end{aligned}$$

Note

In the one variable change of variables formula, we replace dx with $g'(u)$ du because when $x = g(u)$ and g is differentiable, $\Delta x \approx g'(u)\Delta u$. That is, our measure of length changes when we replace the interval $[a, b]$ with the interval $[g(a), g(b)]$.

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Subdivide R into subregions $R_{ij} = T(S_{ij})$.

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where

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Thus, $\int\int_R f(x,y)\,dA$ is approximated by

$$\sum_{i,j} f(g(u_{i-1},v_{j-1}),h(u_{i-1},v_{j-1}))\left|\overrightarrow{T_u(u_{i-1},v_{j-1})} \times \overrightarrow{T_v(u_{i-1},v_{j-1})}\right|\Delta u \Delta v.$$

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Definition

We define the Jacobian of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ by

$$\frac{\partial(x, y)}{\partial(u, v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Theorem

Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type 1 or type 2 plane regions. Suppose that T is one-to-one except perhaps on the boundary of S. Then

$$\int\int_R f(x,y)\ dA = \int\int_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\ dv.$$