

MTHSC 206 SECTION 15.2 – ITERATED INTEGRALS

Kevin James

NOTE

Suppose that $f(x, y)$ is integrable on a rectangle $R = [a, b] \times [c, d]$. Fixing x , we can regard $f(x, y)$ as a function of y only (i.e. we consider the function $g_x(y) = f(x, y)$).

We can thus compute the one variable integral

$$A(x) = \int_c^d g_x(y) dy = \int_c^d f(x, y) dy,$$

which can then be thought of as a function of x .

We can then compute the one variable integral

$$\int_a^b A(x) dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx.$$

DEFINITION

Given a function $f(x, y)$ which is integrable over a rectangle $R = [a, b] \times [c, d]$, the integrals

$$\int_a^b \left[\int_c^d f(x, y) dy \right] dx \quad \text{and} \quad \int_c^d \left[\int_a^b f(x, y) dx \right] dy$$

are called iterated integrals.

EXAMPLE

Compute the two iterated integrals $\int_0^2 \left[\int_1^3 xy \, dy \right] dx$ and $\int_1^3 \left[\int_0^2 xy \, dx \right] dy$

EXAMPLE

Let $R = [0, 2] \times [1, 3]$. Compute $\iint_R xy \, dA$.

THEOREM (FUBINI)

If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\begin{aligned}\iint_R f(x, y) \, dA &= \int_a^b \left[\int_c^d f(x, y) \, dy \right] dx \\ &= \int_c^d \left[\int_a^b f(x, y) \, dx \right] dy\end{aligned}$$

More generally the result holds if f is bounded on R , f is discontinuous only on a finite number of smooth curves and the iterated integrals exist.

EXAMPLE

Let $R = [1, 2] \times [2, 3]$. Compute $\iint_R (x^2 + y) \, dA$.

COROLLARY

Suppose that $f(x, y) = g(x)h(y)$ satisfies the hypotheses of Fubini's theorem on the rectangle $R = [a, b] \times [c, d]$. Then,

$$\int \int_R f(x, y) \, dA = \left(\int_a^b g(x) \, dx \right) \cdot \left(\int_c^d h(y) \, dy \right).$$

EXAMPLE

Let $R = [0, 2] \times [1, 3]$. Compute $\int \int_R xy \, dA$.

EXAMPLE (ORDER OF INTEGRATION MATTERS)

Suppose that $R = [0, \pi] \times [1, 2]$. Compute $\int \int_R x \cos(xy) \, dA$ using both iterated integral formulas.