

MTHSC 206 SECTION 15.3 – DOUBLE INTEGRALS OVER GENERAL REGIONS

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GOAL

Suppose that D is a bounded region in \mathbb{R}^2 and we would like to compute the integral $\int \int_D f(x, y) \, dA$.

SOLUTION

Since D is bounded, there is a rectangle R such that $D \subseteq R$. We define a function $F(x, y)$ on R as

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D, \\ 0 & \text{if } (x, y) \text{ is not in } D. \end{cases}$$

Then we define

$$\int \int_D f(x, y) \, dA = \int \int_R F(x, y) \, dA.$$

DEFINITION

We will consider two types of regions.

TYPE I Regions bounded above and below by a functions and lying above an interval on the x -axis (i.e.

$$D = \{(x, y) \mid a \leq x \leq b; g_1(x) \leq y \leq g_2(x)\}.$$

TYPE II Regions bounded on the left and right by a function and lying beside an interval on the y -axis, (i.e.

$$D = \{(x, y) \mid g_1(y) \leq x \leq g_2(y); c \leq y \leq d\}.$$

NOTE

Some regions will be of both types while some more complex regions are of neither type.

FACT

- ① If $f(x, y)$ is a continuous function on a region $D = \{(x, y) \mid a \leq x \leq b; g_1(x) \leq y \leq g_2(x)\}$ of type I, then

$$\iint_D f(x, y) = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx.$$

- ② If $f(x, y)$ is a continuous function on a region $D = \{(x, y) \mid g_1(y) \leq x \leq g_2(y); c \leq y \leq d\}$ of type II, then

$$\iint_D f(x, y) = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y) dx dy.$$

EXAMPLE

Evaluate $\int \int_D (x + y) \, dA$ where D is the region bounded above and below by the parabolas $y = 3x^2$ and $y = 2x^2 + 4$.

EXAMPLE

Compute $\int \int_D (x^2 + 2y^2) \, dA$ where D is the region bounded by the curves $y = 4x$ and $y = 2x^2$

EXAMPLE

Evaluate the iterated integral $\int_0^1 \int_y^1 \cos(x^2) \, dx \, dy$.

PROPERTIES OF DOUBLE INTEGRALS

- 1 $\int \int_D [f(x, y) + g(x, y)] \, dA = \int \int_D f(x, y) \, dA + \int \int_D g(x, y) \, dA.$
- 2 $\int \int_D cf(x, y) \, dA = c \int \int_D f(x, y) \, dA.$
- 3 If $f(x, y) \geq g(x, y)$ for all $(x, y) \in D$ then $\int \int_D f(x, y) \, dA \geq \int \int_D g(x, y) \, dA.$
- 4 If $D = D_1 \cup D_2$ where $D_1 \cap D_2$ is one dimensional or empty, then $\int \int_D f(x, y) \, dA = \int \int_{D_1} f(x, y) \, dA + \int \int_{D_2} f(x, y) \, dA.$
- 5 $A(D) = \int \int_D 1 \, dA.$
- 6 If $m \leq f(x, y) \leq M$ for all $(x, y) \in D$ then $mA(D) \leq \int \int_D f(x, y) \, dA \leq MA(D).$

EXAMPLE

Estimate $\iint_D e^{\sin(x)\cos(y)} dA$ where D is the circle of radius 3 centered at the origin.