MTHSC 206 Section 15.3 – Double Integrals over General Regions

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$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D,, \\ 0 & \text{if } (x,y) \text{ is not in } D. \end{cases}$$

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Then we define

$$\int\int_D f(x,y)\,\mathrm{dA} = \int\int_R F(x,y)\,\mathrm{dA}.$$

Definition

We will consider two types of regions.

- **Type I** Regions bounded above and below by a functions and lying above an interval on the x-axis (i.e. $D = \{(x, y) \mid a \leq x \leq b; g_1(x) \leq y \leq g_2(x)\}$).
- **Type II** Regions bounded on the left and right by a function and lying beside an interval on the y-axis, (i.e. $D = \{(x, y) \mid g_1(y) \leq x \leq g_2(y); c \leq y \leq d\}$).

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Note

Some regions will be of both types while some more complex regions are of neither type.

Fact

1. *If $f(x,y)$ is a continuous function on a region $D = \{(x,y) \mid a \le x \le b; g_1(x) \le y \le g_2(x)\}$ of type I, then*

$$\int\int_D f(x,y) = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx.$$

2. *If $f(x,y)$ is a continuous function on a region $D = \{(x,y) \mid g_1(y) \le x \le g_2(y); c \le y \le d\}$ of type II, then*

$$\int\int_D f(x,y) = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\, dx\, dy.$$

Example

Evaluate $\int\int_D (x+y)\ dA$ where D is the region bounded above and below by the parabolas $y = 3x^2$ and $y = 2x^2 + 4$.

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Evaluate the iterated integral $\int_0^1 \int_y^1 \cos(x^2)\, dx\, dy$.

Properties of Double Integrals

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4. If $D = D_1 \cup D_2$ where $D_1 \cap D_2$ is one dimensional or empty, then $\int\int_D f(x,y)\ dA = \int\int_{D_1} f(x,y)\ dA + \int\int_{D_2} f(x,y)\ dA.$

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5. $A(D) = \int\int_D 1\ dA.$
6. If $m \leq f(x,y) \leq M$ for all $(x,y) \in D$ then $mA(D) \leq \int\int_D f(x,y)\ dA \leq MA(D).$

Example

Estimate $\int\int_D e^{\sin(x)\cos(y)}\, dA$ where D is the circle of radius 3 centered at the origin.