MTHSC 206 Section 15.4 – Double Integrals in Polar Coordinates

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Recall

The relationship between the Euclidean coordinates (x, y) of a point P and the polar coordinates (r, θ) of the same point P are given by

$$r^2 = x^2 + y^2, \quad x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta).$$

Goal

We will revisit our development of the double integral in terms of polar coordinates.

Suppose that we wish to compute the volume underneath the graph of a function $f(x, y)$ and lying directly above a polar rectangle $R = \{(r, \theta) \mid a \leq r \leq b; \alpha \leq \theta \leq \beta\}$.
We subdivide the polar rectangle into smaller polar rectangles.
Set $\Delta r = \frac{b-a}{n}$ and $\Delta\theta = \frac{\beta-\alpha}{m}$ for some integers m and n.
Let $r_i = a + i\Delta r$ and $\theta_j = \alpha + j\Delta\theta$.
The volume underneath $f(x, y)$ and directly above the rectangle $R_{ij} = \{(r, \theta) \mid r_{i-1} \leq r \leq r_i; \theta_{j-1} \leq \theta \leq \theta_j\}$ can be approximated by

$$f(\bar{r}_i \cos(\bar{\theta}_j), \bar{r}_i \sin(\bar{\theta}_j))\Delta A_{ij},$$

where $\bar{r}_i = \frac{r_i + r_{i-1}}{2}$, $\bar{\theta}_j = \frac{\theta_j - \theta_{j-1}}{2}$ and ΔA_{ij} is the area of R_{ij}.
Recall that
$\Delta A_{ij} = \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta = \frac{1}{2}(r_i + r_{i-1})\Delta r\Delta\theta = \bar{r}_i\Delta r\Delta\theta$.
Thus we approximate the volume underneath R by

$$V \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(\bar{r}_i \cos(\bar{\theta}_j), \bar{r}_i \sin(\bar{\theta}_j))\bar{r}_i\Delta r\Delta\theta.$$

Note

Thus if $f(x, y)$ is continuous, then letting $g(r, \theta) = f(r\cos(\theta), r\sin(\theta))r$, we have that g is integrable and

$$\begin{aligned} V &= \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} g(\bar{r}_i, \bar{\theta}_j)\Delta r \Delta\theta \\ &= \int\int_{[a,b]\times[\alpha,\beta]} g(r, \theta)\ \mathrm{dr}\ \mathrm{d}\theta \\ &= \int_a^b \int_\alpha^\beta g(r, \theta)\ \mathrm{dr}\ \mathrm{d}\theta. \end{aligned}$$

Theorem

If $f(x, y)$ is continuous on a polar rectangle R as above, then

$$\int\int_R f(x, y)\ dA = \int_a^b \int_\alpha^\beta rf(r\cos(\theta), r\sin(\theta))\ dr\ d\theta$$

Example

Evaluate $\int\int_R (3x + 4y^2)\ dA$ where R is the region bounded by the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

Example

Compute the volume of a silo of hight 6 meters and diameter 2 meters whose top is a semi-sphere of radius 1 meter.

Note

As before, we can extend out notion of integration over polar rectangles to integration over more general polar regions.

Fact

Suppose that f is continuous on

$$D = \{(r,\theta) \mid h_1(\theta) \le r \le h_2(\theta); \alpha \le \theta \le \beta\}.$$

Then,

$$\int\int_D f(x,y)\ dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} rf(r\cos(\theta), r\sin(\theta)\ dr\ d\theta.$$

In particular,

$$A(D) = \int\int_D 1\ dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} r\ dr\ d\theta = \int_\alpha^\beta \frac{h_2(\theta)^2 - h_1(\theta)^2}{2}\ d\theta.$$

Example

Compute the area of one petal of the five-petaled flower like region $D = \{(r, \theta) \mid 0 \leq r \leq \cos(5\theta); \frac{-\pi}{10} \leq \theta \leq \frac{\pi}{10}\}$.

Example

Compute the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, inside the cylinder $x^2 + y^2 = 2x$ and above the xy-plane.