

# MTHSC 206 SECTION 15.5 – APPLICATIONS OF DOUBLE INTEGRALS

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## MASS AND DENSITY

Suppose that a lamina represented by a region  $D$  of  $\mathbb{R}^2$  has variable density given by  $\rho(x, y)$ . Then the mass of the lamina can be computed by

$$m = \int \int_D \rho(x, y) \, dA.$$

## ELECTRICAL CHARGE

If an electrical charge is distributed over a region  $D$  of  $\mathbb{R}^2$ , and the charge density (in units of charge per unit area) is given by  $\sigma(x, y)$ , then the total charge is given by

$$Q = \int \int \sigma(x, y) \, dA.$$

### EXAMPLE

Suppose that a charge is distributed over the triangular region  $D = \{(x, y) \mid 0 \leq x \leq 1; 1 - x \leq y \leq 1\}$  with charge density  $\sigma(x, y) = xy \text{ C/m}^2$ . Find the total charge over the region  $D$ .

## MOMENTS AND CENTERS OF MASS

Suppose that a lamina occupies a region  $D$  of  $\mathbb{R}^2$  and has variable density  $\rho(x, y)$ .

Then the mass of a small subrectangle of area  $\Delta A$  is approximately  $\rho(x, y)\Delta A$  where  $(x, y)$  is any point in the rectangle.

Thus, the moment of this subrectangle about the  $x$ -axis is given by  $y\rho(x, y)\Delta A$ .

Thus the total moment of the lamina with respect to the  $x$ -axis is given by

$$M_x = \int \int_D y\rho(x, y) \, dA.$$

Similarly, the total moment of the lamina with respect to the  $y$ -axis is given by

$$M_y = \int \int_D x\rho(x, y) \, dA.$$

## DEFINITION

We define the center of mass (or center of gravity) of a lamina occupying a region  $D$  to be the point  $(\bar{x}, \bar{y})$  such that  $m\bar{x} = M_y$  and  $m\bar{y} = M_x$ , where  $m$  is the total mass of the lamina.

## FACT

The center of mass  $(\bar{x}, \bar{y})$  of a lamina occupying a region  $D$  of  $\mathbb{R}^2$  is given by

$$\bar{x} = \frac{M_y}{m} = \frac{\int \int_D x\rho(x, y) dA}{\int \int_D \rho(x, y) dA}, \quad \bar{y} = \frac{M_x}{m} = \frac{\int \int_D y\rho(x, y) dA}{\int \int_D \rho(x, y) dA}.$$

### EXAMPLE

Suppose that the density at a point  $(x, y)$  of a semicircular lamina is proportional to the distance from  $(x, y)$  to the center of the circle. Find the center of mass for the lamina.

## DEFINITION

Suppose that  $X$  and  $Y$  are random variables. The joint density function of  $X$  and  $Y$  is a function  $f(X, Y)$  such that  $P[(X, Y) \in D] = \int \int_D f(X, Y) \, dA$ .

## NOTE

The joint density function has the following properties.

- 1  $f(X, Y) \geq 0$ .
- 2  $\int \int_{\mathbb{R}^2} f(X, Y) \, dA = 1$ .

## EXAMPLE

Suppose that the joint density function for  $X$  and  $Y$  is given by

$$f(X, Y) = \begin{cases} C(x + y) & \text{if } 0 \leq y \leq 10, \text{ and } 0 \leq x \leq 10, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the constant  $C$ . Then calculate  $P[X \leq 7; Y \geq 2]$ .