MTHSC 206 Section 15.6 – Surface Area

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Surface Area

Suppose that a surface S in $\mathbb{R}^3$ is given as the graph of a function $f(x, y)$ of two variables which has continuous partials f_x and f_y as x and y vary over a rectangle $D = [a, b] \times [c, d]$.
We would like to compute the surface area $A(S)$ of S.
We define $\Delta x = \frac{b-a}{n}$ and $\Delta y = \frac{d-c}{m}$ as usual.
Let $x_i = a + (i-1)\Delta x$, $y_j = c + (j-1)\Delta y$ and let $R_{i,j} = \{(x, y) \mid x_i < x < x_{i+1}; y_j < y, y_{j+1}\}$.
Let $T_{i,j} = \{(x, y, f(x, y)) \mid (x, y) \in R_{i,j}\}$.
We will approximate the area of $T_{i,j}$ by computing the area $\Delta T_{i,j}$ of the parallelogram lying in the tangent plane of S at (x_i, y_j) and lying above $R_{i,j}$.

$$\begin{aligned}\Delta T_{i,j} &= |(\Delta x, 0, f_x(x_i, y_j)\Delta x) \times (0, \Delta y, f_y(x_i, y_j)\Delta y)| \\ &= |(-f_x(x_i, y_j), -f_y(x_i, y_j), 1)\, \Delta x \Delta y| \\ &= \sqrt{f_x(x_i, y_j)^2 + f_y(x_i, y_j)^2 + 1}\Delta x \Delta y.\end{aligned}$$

Definition

If S is a surface in $\mathbb{R}^3$ is given as the graph of a function $f(x, y)$ of two variables which has continuous partials f_x and f_y as x and y vary over a rectangle $D = [a, b] \times [c, d]$, then we define

$$\begin{aligned} A(S) &= \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} \sqrt{f_x(x_i, y_j)^2 + f_y(x_i, y_j)^2 + 1}\Delta x \Delta y \\ &= \int\int_D \sqrt{f_x(x, y)^2 + f_y(x, y)^2 + 1}\ \mathrm{dA}. \end{aligned}$$

Example

Compute the area of the part of the paraboloid $z = x^2 + y^2$ lying below the plane $z = 4$.