

# MTHSC 206 SECTION 15.7 – TRIPLE INTEGRALS

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## DEFINITION

The triple integral of  $f(x, y, z)$  over the box

$B = \{(x, y, z) \mid a \leq x \leq b; c \leq y \leq d; r \leq z \leq s\}$  is defined by

$$\int \int \int_B f(x, y, z) \, dV = \lim_{l, m, n \rightarrow \infty} \sum_{i=0}^l \sum_{j=0}^m \sum_{k=0}^n f(x_i, y_j, z_k) \Delta V.$$

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## THEOREM (FUBINI)

If  $f(x, y, z)$  is continuous on  $B$  then,

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## NOTE

There are a total of six possible orders of integration for triple integrals. A more general version of Fubini's theorem guarantees that these all have the same value.

### EXAMPLE

Evaluate the triple integral  $\int \int \int_B xyz \, dV$  where  $B = [0, 1] \times [0, 1] \times [0, 1]$ .

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We consider three types of regions in  $\mathbb{R}^3$ .

**TYPE 1**  $E = \{(x, y, z) \mid (x, y) \in D; u_1(x, y) \leq z \leq u_2(x, y)\}$

**TYPE 2**  $E = \{(x, y, z) \mid (y, z) \in D; u_1(y, z) \leq x \leq u_2(y, z)\}$

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## FACT

If  $E$  is a type 1, 2 or 3 region, then  $\int \int \int_E f(x, y, z) dV$  has the same value as

$$\left\{ \begin{array}{l} \int \int_D \left[ \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) dz \right] dA \quad \text{if } E \text{ is type 1,} \\ \int \int_D \left[ \int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z) dx \right] dA \quad \text{if } E \text{ is type 2,} \\ \int \int_D \left[ \int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z) dy \right] dA \quad \text{if } E \text{ is type 3.} \end{array} \right.$$

### EXAMPLE

Evaluate  $\int \int \int_E z \, dV$  where  $E$  is the solid tetrahedron bounded by the four planes  $x = 0$ ,  $y = 0$ ,  $z = 0$  and  $x + y + z = 1$ .



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### EXAMPLE

Evaluate  $\int \int \int_E \sqrt{y^2 + z^2} \, dV$ , where  $E$  is the solid bounded by the paraboloid  $x = y^2 + z^2$  and the plane  $x = 4$ .

## VOLUME

If  $E$  is a solid region in  $\mathbb{R}^3$ , then  $V(E) = \int \int \int_E dV$ .

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## MASS AND DENSITY

Suppose that  $E$  is a solid in  $\mathbb{R}^3$  with density function  $\rho(x, y, z)$ . Then the mass of  $E$  is given by  $m = \int \int \int \rho(x, y, z) dV$ .

## MOMENTS AND CENTER OF MASS

Suppose that  $E$  is a solid in  $\mathbb{R}^3$  with density function  $\rho(x, y, z)$ . The moments of  $E$  about the three coordinate planes are

$$M_{yz} = \int \int \int_E x \rho(x, y, z) \, dV, \quad M_{xz} = \int \int \int_E y \rho(x, y, z) \, dV,$$
$$M_{xy} = \int \int \int_E z \rho(x, y, z) \, dV.$$

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Thus the center of mass of  $E$  is at  $(\bar{x}, \bar{y}, \bar{z})$  where

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m} \quad \text{and} \quad \bar{z} = \frac{M_{xy}}{m}$$

### EXAMPLE

Find the center of mass of the solid of constant density  $\rho$  that is bounded by the parabolic cylinder  $y = x^2$  and the planes  $y = 4$ ,  $z = 0$  and  $z = 1$ .