MTHSC 206 Section 15.8 – Triple Integrals in Cylindrical Coordinates

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$$\begin{aligned} x &= r\cos(\theta), \quad y = r\sin(\theta), \quad z = z. \\ r^2 &= x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z. \end{aligned}$$

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Example

Describe the surface whose cylindrical coordinates satisfy $z = r$.

Fact

Suppose that $E = \{(x,y,z) \mid (x,y) \in D; u_1(x,y) \le z \le u_2(x,y)\}$ *where* $D = \{(r,\theta) \mid \alpha \le \theta \le \beta; h_1(\theta) \le r \le h_2(\theta)\}$. *Then*

$$\int\int\int_E f(x,y,z)\,dV = \int\int_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right]\,dA$$

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Example

Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}\left(x^2+y^2\right)\ \mathrm{dz}\ \mathrm{dy}\ \mathrm{dx}$.