MTHSC 206 Section 16.3 – The Fundamental Theorem of Line Integrals

Kevin James

Theorem

Let C be a smooth curve parametrized by $r(t)$ for $a \leq t \leq b$. Let f be a differentiable function whose gradient ∇f is continuous on C. Then

$$\int_C \nabla f \cdot \, dr = f(r(b)) - f(r(a)).$$

Proof.

Let $g(t) = f(r(t))$. Then g is a real valued function of one variable and $g'(t) = \nabla f \cdot r'(t)$. So the theorem follows from the fundamental theorem of calculus. □

Definition

Suppose that F is a continuous vector field with domain D. We say that $\int_C F \cdot \, dr$ is independent of path if $\int_{C_1} F \cdot \, dr = \int_{C_2} F \cdot \, dr$ for any two two paths C_1 and C_2 in D with the same initial and ending points.

Note

We saw last time that not all vector fields are independent of path.

Definition

A curve C whose initial point and ending point are the same is called a closed curve.

Theorem

$\int_C F \cdot\ dr$ is independent of path in D if and only if $\int_C F \cdot\ dr = 0$ for all closed curves.

Proof.

Suppose that C_1 is a curve with initial point A and ending point B and that C_2 is a curve with initial point B and ending point A.
Then C_1 and $-C_2$ have the same initial and ending points and $C = C_1 \cup C_2$ is a closed curve.
Further, we have

$$\int_C F \cdot\ \mathrm{dr} = \int_{C_1} F \cdot\ \mathrm{dr} + \int_{C_2} F \cdot\ \mathrm{dr} = \int_{C_1} F \cdot\ \mathrm{dr} - \int_{-C_2} F \cdot\ \mathrm{dr}.$$

$\square$

Theorem

Suppose D is an open connected region and that F is a vector field on D. If $\int_C F \cdot$ dr is independent of path then F is conservative.

Idea of proof.

Take $f(x, y) = \int_{(a,b)}^{(x,y)} F \cdot$ dr where (a, b) is any point of D. Note that since D is connected, there is a path C from (a, b) to (x, y) for any point $(x, y) \in D$. Since, $\int_C F \cdot$ dr is path independent, it does not matter which path we choose. □

Theorem

If $F(x,y) = [P(x,y), Q(x,y)]$ is a conservative vector field (-i.e. $F = \nabla f$ for some $f(x,y)$.) where P and Q have continuous first order partial derivatives on D then throughout D we have

$$P_y = Q_x.$$

Proof.

This follows from Clairaut's theorem. □

Definition

A simply connected region D is a region in which every simple closed curve encloses only points of D.

Theorem

Let $F = [P, Q]$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first order partial derivatives and $P_y = Q_x$ throughout D. Then F is conservative.

Example

Note that the vector field $F(x, y) = [(x - y), (x - 2)]$ is not conservative.

Example

Determine if the vector field $F(x, y) = [(3 + 2xy), (x^2 - 3y^2)]$ is conservative.

Example

Find the potential function for the field in the previous example.

Suppose that a continuous force field F moves a particle along a curve C which is parametrized by $r(t)$ with $a \leq t \leq b$.
Recall that $F = ma = mr''(t)$. So the work done is

$$\begin{aligned} W &= \int_C F \cdot \, \mathrm{dr} = \int_a^b F(r(t)) \cdot r'(t) \, \mathrm{dt} = \int_a^b mr''(t) \cdot r'(t) \, \mathrm{dt} \\ &= \frac{m}{2} \int_a^b \frac{\mathrm{d}}{\mathrm{dt}} [r'(t) \cdot r'(t)] \, \mathrm{dt} = \frac{m}{2} \int_a^b \frac{\mathrm{d}}{\mathrm{dt}} [|r'(t)|^2] \, \mathrm{dt} \\ &= \frac{m}{2} \left(|r'(b)|^2 - |r'(a)|^2 \right) = \frac{m}{2} |v(b)|^2 - \frac{m}{2} |v(a)|^2. \end{aligned}$$

Physicists define the kinetic energy of a particle at $r(c)$ as $K(r(c)) = \frac{m}{2} |v(c)|^2$, which gives $W = K(r(b)) - K(r(a))$.

If F is conservative with potential function f, then we define the potential energy of the particle as
$P(r(c)) = -f(r(c)) \Rightarrow F(r(c)) = \nabla f(r(c)) = -\nabla P(r(c))$.
So we have
$K(r(b)) - K(r(a)) = W = \int_C F \cdot \, \mathrm{dr} = -\int_C \nabla P \cdot \, \mathrm{dr} =$
$P(r(a)) - P(r(b))$, which implies that
$K(r(b)) + P(r(b)) = K(r(a)) + P(r(a))$.