MTHSC 206 SECTION 16.3 – THE FUNDAMENTAL THEOREM OF LINE INTEGRALS

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Theorem

Let C be a smooth curve parametrized by r(t) for $a \le t \le b$. Let f be a differentiable function whose gradient ∇f is continuous on C. Then

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a)).$$

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PROOF.

Let g(t) = f(r(t)). Then g is a real valued function of one variable and $g'(t) = \nabla f \cdot r'(t)$. So the theorem follows from the fundamental theorem of calculus.

Suppose that F is a continuous vector field with domain D. We say that $\int_C F \cdot d\mathbf{r}$ is independent of path if $\int_{C_1} F \cdot d\mathbf{r} = \int_{C_2} F \cdot d\mathbf{r}$ for any two two paths C_1 and C_2 in D with the same initial and ending points.

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DEFINITION

A curve *C* whose initial point and ending point are the same is called a closed curve.

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Proof.

Suppose that C_1 is a curve with initial point A and ending point B and that C_2 is a curve with initial point B and ending point A. Then C_1 and $-C_2$ have the same initial and ending points and $C = C_1 \cup C_2$ is a closed curve.

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Further, we have

$$\int_C F \cdot \ \mathrm{d} r = \int_{C_1} F \cdot \ \mathrm{d} r + \int_{C_2} F \cdot \ \mathrm{d} r = \int_{C_1} F \cdot \ \mathrm{d} r - \int_{-C_2} F \cdot \ \mathrm{d} r.$$

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IDEA OF PROOF.

Take $f(x,y) = \int_{(a,b)}^{(x,y)} F \cdot dr$ where (a,b) is any point of D. Note that since D is connected, there is a path C from (a,b) to (x,y) for any point $(x,y) \in D$. Since, $\int_C F \cdot dr$ is path independent, it does not matter which path we choose.

If F(x,y) = [P(x,y), Q(x,y)] is a conservative vector field (-i.e. $F = \nabla f$ for some f(x,y).) where P and Q have continuous first order partial derivatives on D then throughout D we have

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Proof.

This follows from Clairaut's theorem.



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THEOREM

Let F = [P, Q] be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first order partial derivatives and $P_v = Q_x$ throughout D. Then F is conservative.

EXAMPLE

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EXAMPLE

Find the potential function for the field in the previous example.

Suppose that a continuous force field F moves a particle along a curve C which is parametrized by r(t) with $a \le t \le b$.

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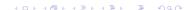
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$$K(r(b)) + P(r(b)) = K(r(a)) + P(r(a)).$$