MTHSC 206 Section 16.3 – The Fundamental Theorem of Line Integrals

Kevin James

Theorem

Let C be a smooth curve parametrized by $r(t)$ for $a \leq t \leq b$. Let f be a differentiable function whose gradient ∇f is continuous on C. Then

$$\int_C \nabla f \cdot \, dr = f(r(b)) - f(r(a)).$$

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Proof.

Let $g(t) = f(r(t))$. Then g is a real valued function of one variable and $g'(t) = \nabla f \cdot r'(t)$. So the theorem follows from the fundamental theorem of calculus. $\square$

Definition

Suppose that F is a continuous vector field with domain D. We say that $\int_C F \cdot \, dr$ is independent of path if $\int_{C_1} F \cdot \, dr = \int_{C_2} F \cdot \, dr$ for any two two paths C_1 and C_2 in D with the same initial and ending points.

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A curve C whose initial point and ending point are the same is called a closed curve.

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Suppose that C_1 is a curve with initial point A and ending point B and that C_2 is a curve with initial point B and ending point A. Then C_1 and $-C_2$ have the same initial and ending points and $C = C_1 \cup C_2$ is a closed curve.

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Then C_1 and $-C_2$ have the same initial and ending points and $C = C_1 \cup C_2$ is a closed curve.
Further, we have

$$\int_C F \cdot\ \mathrm{dr} = \int_{C_1} F \cdot\ \mathrm{dr} + \int_{C_2} F \cdot\ \mathrm{dr} = \int_{C_1} F \cdot\ \mathrm{dr} - \int_{-C_2} F \cdot\ \mathrm{dr}.$$

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Idea of proof.

Take $f(x,y) = \int_{(a,b)}^{(x,y)} F \cdot \, dr$ where (a,b) is any point of D. Note that since D is connected, there is a path C from (a,b) to (x,y) for any point $(x,y) \in D$. Since, $\int_C F \cdot \, dr$ is path independent, it does not matter which path we choose. $\square$

Theorem

If $F(x,y) = [P(x,y), Q(x,y)]$ *is a conservative vector field (-i.e.* $F = \nabla f$ *for some* $f(x,y)$*.) where* P *and* Q *have continuous first order partial derivatives on* D *then throughout* D *we have*

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This follows from Clairaut's theorem. □

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Theorem

Let $F = [P, Q]$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first order partial derivatives and $P_y = Q_x$ throughout D. Then F is conservative.

Example

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Find the potential function for the field in the previous example.

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&= \frac{m}{2} \int_a^b \frac{\mathrm{d}}{\mathrm{dt}} [r'(t) \cdot r'(t)] \, \mathrm{dt} = \frac{m}{2} \int_a^b \frac{\mathrm{d}}{\mathrm{dt}} [|r'(t)|^2] \, \mathrm{dt} \\
&= \frac{m}{2} \left(|r'(b)|^2 - |r'(a)|^2 \right) = \frac{m}{2} |v(b)|^2 - \frac{m}{2} |v(a)|^2.
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$P(r(a)) - P(r(b))$, which implies that
$K(r(b)) + P(r(b)) = K(r(a)) + P(r(a))$.