MTHSC 206 Section 16.4 – Green's Theorem

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Theorem

Let C be a positively oriented, piecewise smooth, simple closed curve in $\mathbb{R}^2$. Let D be the region bounded by C. If $P(x,y)($ and $Q(x,y)$ have continuous partial derivatives on an open region containing D, then

$$\int_C P\ dx + Q\ dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA.$$

Note

In the situation above, we sometimes denote C as ∂D.

Sketch of Proof

We will assume that the region D is both of type I and type II.
Note that it is enough to show that
$\int_C P\ \mathrm{dx} = -\int\int_D \frac{\partial P}{\partial y}\ \mathrm{dA}$, and $\int_C Q\ \mathrm{dy} = \int\int_D \frac{\partial Q}{\partial x}\ \mathrm{dA}$.
We will show that first equality.
Writing

$$D = \{(x, y) \mid a \le x \le b; g_1(x) \le y \le g_2(x)\},$$

we have

$$\begin{aligned} \int\int_D \frac{\partial P}{\partial y}\ \mathrm{dA} &= \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\ \mathrm{dy\ dx} \\ &= \int_a^b [P(x, g_2(x)) - P(x, g_1(x))]\ \mathrm{dx}. \end{aligned}$$

Sketch of proof continued ...

Now we break C into 4 curves $C1$, $C2$, $C3$ and $C4$ given by:

$$\begin{aligned}
C1 &: [t, g_1(t)]; \quad a \le t \le b \\
C2 &: [b, t]; \quad g_1(b) \le t \le g_2(b) \\
-C3 &: [t, g_2(t)]; \quad a \le t \le b \\
-C4 &: [a, t]; \quad g_1(a) \le t \le g_2(a).
\end{aligned}$$

Note that $\int_{C2} P \text{ dx} = 0 = \int_{C4} P \text{ dx}$. Thus,

$$\begin{aligned}
\int_C P \text{ dx} &= \int_{C1} P \text{ dx} - \int_{-C3} P \text{ dx} \\
&= \int_a^b [P(t, g_1(t)) - P(t, g_2(t))] \text{ dt} \\
&= -\int\int_D \frac{\partial P}{\partial y} \text{ dA}.
\end{aligned}$$

Example

Evaluate $\int_C x^3 \, \mathrm{dx} + xy \, \mathrm{dy}$ where C is the curve bounding the triangular region with vertices $(0,0)$, $(1,0)$ and $(0,2)$.

Example

Evaluate $\int_C (3y - e^{\sin(x)}) \, \mathrm{dx} + (7x + \sqrt{y^4 + 1}) \, \mathrm{dy}$ where C is the circle about the origin of radius 3.

Note

If P and Q are known to be zero on C and if D is the interior of C then no matter the behavior of P and Q in D, we have $\int \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \int_C P \, \mathrm{dx} + Q \, \mathrm{dy} = 0$.

Note

We can use Green's theorem to calculate area. We simply need to arrange for $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$.
Here are some possible choices:

$$\begin{array}{lll} P(x,y) = 0 & P(x,y) = -y, & P(x,y) = \dfrac{-y}{2} \\ Q(x,y) = x & Q(x,y) = 0, & Q(x,y) = \dfrac{x}{2}. \end{array}$$

Then, Green's theorem gives

$$A = \int_C x \text{ dy} = -\int_C y \text{ dx} = \frac{1}{2}\int_C x \text{ dy} - y \text{ dx}.$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2$.

Note

We can use Green's theorem to integrate over regions which are not of type I and of type II but are finite unions of such regions.

Example

Evaluate $\int_C y^2 \, \mathrm{dx} + 3xy \, \mathrm{dy}$ where C is the boundary of the region bounded above by the upper semicircle of radius 2 and below by the upper semicircle of radius 1.

Note

With some care, Green's theorem can be extended to regions with holes.

Example

If $F(x,y) = \left[\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right]$, show that $\int_C F \cdot\ \mathrm{dr} = 2\pi$ for every positively oriented, simple, closed path that encloses the origin.