MTHSC 206 Section 16.4 – Green's Theorem

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Theorem

Let C be a positively oriented, piecewise smooth, simple closed curve in $\mathbb{R}^2$. Let D be the region bounded by C. If $P(x, y)($ and $Q(x, y)$ have continuous partial derivatives on an open region containing D, then

$$\int_C P\ dx + Q\ dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA.$$

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Note

In the situation above, we sometimes denote C as ∂D.

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$$D = \{(x,y) \mid a \le x \le b; g_1(x) \le y \le g_2(x)\},$$

we have

$$\begin{aligned}\int\int_D \frac{\partial P}{\partial y}\ \mathrm{dA} &= \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\ \mathrm{dy\ dx} \\ &= \end{aligned}$$

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Sketch of proof continued ...

Now we break C into 4 curves $C1$, $C2$, $C3$ and $C4$ given by:

$$\begin{aligned} C1 &: [t, g_1(t)]; \quad a \le t \le b \\ C2 &: [b, t]; \quad g_1(b) \le t \le g_2(b) \\ -C3 &: [t, g_2(t)]; \quad a \le t \le b \\ -C4 &: [a, t]; \quad g_1(a) \le t \le g_2(a). \end{aligned}$$

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Example

Evaluate $\int_C x^3 \, \mathrm{d}x + xy \, \mathrm{d}y$ where C is the curve bounding the triangular region with vertices $(0, 0)$, $(1, 0)$ and $(0, 2)$.

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Note

If P and Q are known to be zero on C and if D is the interior of C then no matter the behavior of P and Q in D, we have $\int \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \int_C P \, \mathrm{dx} + Q \, \mathrm{dy} = 0$.

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Here are some possible choices:

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Then, Green's theorem gives

$$A = \int_C x \text{ dy} = -\int_C y \text{ dx} = \frac{1}{2}\int_C x \text{ dy} - y \text{ dx}.$$

Example

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Example

Evaluate $\int_C y^2 \, \mathrm{dx} + 3xy \, \mathrm{dy}$ where C is the boundary of the region bounded above by the upper semicircle of radius 2 and below by the upper semicircle of radius 1.

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Example

Use the extended version of Green's theorem mentioned above to show that if $F(x, y) = \left[\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right]$, then $\int_C F \cdot \, \mathrm{dr} = 2\pi$ for every positively oriented, simple, closed path that encloses the origin.