MTHSC 206 Section 16.5 – Curl and Divergence

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Definition

Suppose that $F = [P, Q, R]$ is a vector field on $\mathbb{R}^3$. We define the curl of F as

$$\begin{aligned}
\operatorname{curl}(F) &= \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right), \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right), \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] \\
&= \det \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix} \\
&= \nabla \times F,
\end{aligned}$$

where ∇ denotes the operator $\nabla = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}]$.

Example

Let $F = [x^2 z, xyz, z^2]$. Find $\operatorname{curl}(F)$.

Theorem

If $f(x, y, z)$ has continuous second order partials, then $curl(\nabla f) = 0$.

Corollary

Suppose that $F = [P, Q, R]$ is a vector field on $\mathbb{R}^3$ with $curl(F) \neq 0$. Then F is not conservative.

Example

Is the vector field $F(x, y, z) = [x^2 z, xyz, -y^2]$ conservative?

Theorem

Suppose that F is a vector field on $\mathbb{R}^3$ all of whose partial derivatives are continuous and that $curl(F) = 0$. Then, F is conservative. (Actually, we only need the domain of F to be simply connected for the result to hold).

Example

Let $F(x, y, z) = [yz + 1, xz + 1, xy + 2z]$. Show that F is conservative and find its potential function.

Definition

Suppose that $F(x, y, z) = [P, Q, R]$ is a vector field on $\mathbb{R}^3$. We define the divergence of F as

$$\begin{aligned} \operatorname{div}(F) &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \\ &= \nabla \cdot F. \end{aligned}$$

Example

Let $F(x, y, z) = [xz, xyz, -y^2]$. Find $\operatorname{div}(F)$.

Theorem

Suppose that $F(x, y, z) = [P, Q, R]$ *is a vector field on* $\mathbb{R}^3$ *and that* P, Q, *and* R *all have continuous second order partials. Then,*

$$div(\ curl(F)) = 0.$$

Corollary

Suppose that $F(x, y, z) = [P, Q, R]$ *is a vector field on* $\mathbb{R}^3$ *and* $div(F) \neq 0$. *Then* F *is not the curl of another vector field.*

Example

Show that $F(x, y, z) = [xz, xyz, -y^2]$ is not the curl of another vector field.

Theorem

Suppose that $F(x,y) = [P,Q]$ is a vector field on $\mathbb{R}^2$. We can think of F as a vector field on $\mathbb{R}^3$ defined by $F(x,y,z) = [P(x,y), Q(x,y), 0]$. With this interpretation we have

$$\int_C F \cdot \ dr = \int\int_D curl(F) \cdot k \ dA$$

$$\int_C F \cdot n \ ds = \int\int_D div(F)(x,y) \ dA$$

where C is the boundary curve of the region D, $r(t)$ parametrizes C and n is the normal vector to r, and we assume that D and C satisfy the hypotheses of Green's theorem.