MTHSC 206 Section 16.6 – Parametric Surfaces and their Areas

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Note

As we parametrized space curves with a vector valued function $r(t)$ of one variable, we can parametrize a surface (-i.e. a two dimensional object in $\mathbb{R}^3$) with a vector valued function $r(u, v)$ of two variables.

If there is a vector valued function $r(u, v) = [x(u, v), y(u, v), z(u, v)]$ such that the surface S is traced out by r as (u, v) varies over some region D, then we call S a parametric surface.

We refer to the equations $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$ as the parametric equations of the surface.

Example

Identify the surface parametrized by
$r(u, v) = [2\cos(u), v, 2\sin(u)]$.

Note

In order to get an idea of the graph of a parametric surface, it is often useful to hold either u or v constant and let the other vary. The functions $r(u, v_0)$ and $r(u_0, v)$ where u_0 and v_0 are constants are vector valued functions of one variable and thus trace out space curves. These curves are referred to as <u>grid curves</u>.

Example

What are the grid curves in the above example?

Example

Find a parametrization of the plane passing through the point P_0 with position vector r_0 and containing two nonparallel vectors $\vec{a}$ and $\vec{b}$.

Example

Find a parametric representation of the sphere $x^2 + y^2 + z^2 = a^2$.

Note

Parametrizations of a surface are not unique.

Example

Find two parametrizations of the cone $z = 2\sqrt{x^2 + y^2}$.

Fact

If S is the surface formed by rotating the graph of $y = f(x)$ above $[a, b]$ about the x-axis, then we can parametrize S by

$$r(x, \theta) = [x, f(x)\cos(\theta), f(x)\sin(\theta)], \quad a \leq x \leq b; \quad 0 \leq \theta \leq 2\pi.$$

Example

Find the parametric equations for the surface generated by rotating the curve $y = x^2$, $0 \leq x \leq 2$ about the x-axis.

Definition

Suppose that S is a surface parametrized by $r(u,v)$, $(u,v) \in D$.
We say that S is smooth if $r_u = [\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}]$ and $r_v = [\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}]$ satisfy $r_u \times r_v \neq 0$ for all $(u,v) \in D$.

Fact

If S is a smooth surface parametrized by $r(u,v)$, then the tangent plane to S at the point $r(u_0,v_0)$ is the plane containing the point $r(u_0,v_0)$ and the vectors $r_u(u_0,v_0)$ and $r_v(u_0,v_0)$.
This plane has normal vector $n = r_u(u_0,v_0) \times r_v(u_0,v_0)$.

Example

Find the tangent plane to the surface parametrized by
$r(u,v) = [u^2, v^2, u+2v]$ at $(1,1,3)$.

Definition

Suppose that a smooth surface S is parametrized by $r(u, v) = [x(u, v), y(u, v), z(u, v)]$, $(u, v) \in D$ and that S is covered just once as (u, v) traverses D. Then the surface area of S is

$$A(S) = \int\int_D |r_u \times r_v| \, \mathrm{dA}.$$

Example

Find the surface area of the sphere of radius a.

Definition

Suppose that S is the graph of a function $f(x, y)$ then S is parametrized by $r(u, v) = [u, v, f(u, v)]$ and the surface area of S is

$$A(S) = \int\int_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, \mathrm{dA}.$$

Example

Find the area of the part of the paraboloid $z = x^2 + y^2$ which lies below the plane $z = 9$.