MTHSC 206 Section 13.2 –Vectors

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Geometric Description of Vectors

Definition

1. A vector $v \in \mathbb{R}^3$ is a pair of points $v = \vec{AB}$. The point A is said to be the initial point and the point B the terminal point.
2. The magnitude of $v = \vec{AB}$ is defined to be $|AB|$.
3. Two vectors are considered equal if they have the same direction (i.e. they lie on parallel lines and have the same orientation) and the same magnitude.
4. The zero vector denoted by 0 is the vector whose initial and terminal points are the same. This vector has magnitude 0 and has no associated direction.

Definition (Addition)

If u and v are vectors positioned so that the initial point of v is at the terminal point of u, then $u + v$ is the vector with initial point the same as the initial point of u and terminal point the same as the terminal point of v.

Note

The parallelogram law assures us that $u + v = v + u$.

Definition (Scalar multiplication)

If $c \in \mathbb{R}$ and $v \in \mathbb{R}^3$ then the scalar multiple cv of v by c is the vector whose length is $|c|$ times the length of v and whose direction is the same as v if $c > 0$ and is opposite of the direction of v if $c < 0$. If $c = 0$ or $v = 0$ then $cv = 0$.

Note

1. We define the negative of v as $(-1)v$.
2. We define the difference of two vectors as $u - v = u + (-1)v$.

Convention

1. Given a vector $a = \vec{AB}$, we can associate to it the pair (a_1, a_2) if we are in 2 dimensions or the 3-tuple (a_1, a_2, a_3) if we are in 3 dimensions where one can move from the initial point A to the terminal point B by moving a_1 units in the x direction, a_2 units in the y direction (and a_3 units in the z direction).
2. The pair or 3-tuple above is called an algebraic vector.
3. This completely characterizes the vector v.
4. So, we typically think of vectors as having their initial point at the origin and their terminal point at (a_1, a_2, a_3).
5. We say that any geometric vector $v = \vec{AB}$ with associated algebraic vector $a = (a_1, a_2, a_3)$ is a representation of a.

Fact

Given the points $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$, the algebraic vector a with geometric representation $\vec{AB}$ is

$$a = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

Fact

The magnitude or length of an algebraic vector a (or of any of its geometric representations) is given by

$$|a| = \begin{cases} \sqrt{a_1^2 + a_2^2} & \text{in 2 dimensions,} \\ \sqrt{a_1^2 + a_2^2 + a_3^2} & \text{in 3 dimensions.} \end{cases}$$

Fact

If $a = (a_1, a_2, a_3)$ *and* $b = (b_1, b_2, b_3)$ *and* $c \in \mathbb{R}$, *then*

$$\begin{aligned} a + b &= (a_1 + b_1, a_2 + b_2, a_3 + b_3), \\ a - b &= (a_1 - b_1, a_2 - b_2, a_3 - b_3) \quad \textit{and} \\ ca &= (ca_1, ca_2, ca_3). \end{aligned}$$

Exercise

Suppose that $a = (1, 2, -1)$ and $b = (-1, 3, 2)$. Find $|a|$, $a + b$ and $2a + 3b$.

Properties of Vectors

If a, b and c are vectors $d, e \in \mathbb{R}$, then,

1. $a + b = b + a$.
2. $(a + b) + c = a + (b + c)$.
3. $a + 0 = a$.
4. $a + (-a) = 0$.
5. $d(a + b) = da + db$.
6. $(e + d)a = ea + ea$.
7. $(ed)a = e(da)$.
8. $(1)a = a$.

Definition

The standard basis vectors are

$$i = (1, 0, 0), \quad j = (0, 1, 0) \quad \text{and} \quad k = (0, 0, 1).$$

Fact

$(a_1, a_2, a_3) = a_1 i + a_2 j + a_3 k.$

Exercise

Suppose that $a = 2i - 3j + k$ and $b = i - j + 3k$. Express the vector $2a + b$ in terms of i, j and k.

Definition

A unit vector is a vector whose length is 1.

Example

i, j and k are unit vectors.

Normalization

If $0 \neq a \in \mathbb{R}^3$, then the unit vector which points in the same direction as a is given by $\frac{1}{|a|}a$.

Exercise

Suppose that a 100-lb weight hangs from two wires which are at 60° and 30° to the flat ceiling. Find the tension forces T_1 and T_2 in both wires and their magnitudes. *(**Hint:** Treat all forces as vectors in $\mathbb{R}^2$ expressed in terms of i and j.)*