

# MTHSC 206 SECTION 13.5 –EQUATIONS OF LINES AND PLANES

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## DEFINITION

A line in  $\mathbb{R}^3$  can be described by a point and a direction vector. Given the point  $r_0$  and the direction vector  $v$ . Any point  $r$  on the line through  $r_0$  and parallel to  $v$  satisfies  $r = r_0 + tv$  for some  $t \in \mathbb{R}$ .

## DEFINITION

If we let  $r_0 = (x_0, y_0, z_0)$  and  $v = (a, b, c)$ , then the line described above could also be described by the parametric equations

$$x = x_0 + at, \quad y = y_0 + bt \quad \text{and} \quad z = z_0 + tc.$$

## DEFINITION

If the coordinates of the direction vector are all non zero, then we have a third description of the line given by solving each of the parametric equations for  $t$  and equating, namely

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

## EXERCISE

Give vector, parametric and symmetric equations for the line passing through the points  $A = (1, 1, 1)$  and  $B = (0, -1, 2)$ . At what point does this line intersect the  $xy$ -plane?

## DEFINITION

A plane in  $\mathbb{R}^3$  can be described by a point  $r_0$  in the plane along with a normal vector  $n$  which is orthogonal to all vectors in the plane. Given this information, we see that  $r$  is in the plane if and only if  $r - r_0$  is orthogonal to  $n$ , that is

$$\begin{aligned}n \cdot (r - r_0) &= 0, \quad \text{or} \\n \cdot r &= n \cdot r_0.\end{aligned}$$

Either of these equations is called the vector equation of the plane.

## NOTE

Letting  $n = (a, b, c)$ ,  $r = (x, y, z)$  and  $r_0 = (x_0, y_0, z_0)$ , we obtain the scalar equation of the plane through the point  $r_0 = (x_0, y_0, z_0)$  with normal vector  $n = (a, b, c)$ , namely

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

## NOTE

The last equation can be rewritten in the form

$$ax + by + cz + d = 0.$$

This is called a linear equation of the plane.

## EXERCISE

Find an equation for the plane containing the points  $(1, 0, 0)$ ,  $(0, 1, 0)$  and  $(0, 0, 1)$ .

## FACT

- 1 *Two planes are parallel if and only if their normal vectors are parallel.*
- 2 *The angle between two intersecting planes is equal to the angle between their normal vectors.*

## EXERCISE

Find the angle between the planes  $x + y + z = 0$  and  $x + 2y + 3z = 0$ . Describe their intersection.

## FACT

The distance from the point  $P = (x_1, y_1, z_1)$  to the plane  $ax + by + cz + d = 0$  is given by

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

## PROOF.

Note that a normal vector to the plane is  $n = (a, b, c)$ .

Let  $P_0$  be any point in the plane and let

$$b = \vec{P_0P} = (x_1 - x_0, y_1 - y_0, z_1 - z_0).$$

$$\text{Then, } D = |\text{comp}_n(b)| = \frac{|n \cdot b|}{|n|} = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}.$$

Now note that  $-ax_0 - by_0 - cz_0 = d$ , since  $P_0$  is in the plane.  $\square$