MTHSC 206 Section 14.2 – Derivatives and Integrals of Vector Functions

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DEFINITION

Suppose that $r(t)$ is a vector function. We define its derivative by

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r'(t) = \lim_{h \to 0} \left[\frac{r(t+h) - r(t)}{h} \right]$$

NOTE

When $r'(t_0)$ exists and is nonzero, we refer to it as the tangent vector to the space curve defined by $r(t)$ at the point $P = r(t_0)$. In this case, the line tangent to the curve at $P = r(t_0)$ is given by $P + r'(t_0)t$.

DEFINITION

The unit tangent vector to the curve is defined by

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

THEOREM

Suppose that $r(t) = (f(t), g(t), h(t)) = f(t)i + g(t)j + h(t)k$, where f, g and h are differentiable functions. Then,

$$r'(t) = (f'(t), g'(t), h'(t)) = f'(t)i + g'(t)j + h'(t)k.$$

EXAMPLE

Find the derivative of $r(t) = (\cos(t), \sin(t), 2t)$. Find the unit tangent vector at the point where $t = \pi$.

Theorem (Properties of Derivatives)

Suppose that u, v are differentiable vector functions, $c \in \mathbb{R}$ and $f(t)$ is a real valued function. Then,

1. $\frac{d}{dt}[u(t) + v(t)] = u'(t) + v'(t)$.

2. $\frac{d}{dt}[cu(t)] = cu'(t)$.

3. $\frac{d}{dt}[f(t)u(t)] = f'(t)u(t) + f(t)u'(t)$.

4. $\frac{d}{dt}[u(t) \cdot v(t)] = u'(t) \cdot v(t) + u(t) \cdot v'(t)$.

5. $\frac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t)$.

6. $\frac{d}{dt}[u(f(t))] = f'(t)u'(f(t))$.

EXAMPLE

Show that if $|r(t)| = c$ is constant, then $r'(t)$ is orthogonal to $r(t)$ for all t.

EXAMPLE

What is the derivative of $r(t) = (\cos(\sin(t)), \sin(\sin(t)), e^{\sin(t)})$?

DEFINITION

We define the definite integral of a continuous vector function
$r(t) = (f(t), \overline{g(t), h(t)})$ on an interval $[a, b]$ as

$$\int_a^b r(t)dt = \lim_{n \to \infty} \sum_{i=1}^n r(t_i*)\Delta t$$

$$= \left(\lim_{n \to \infty} \sum_{i=1}^n f(t_i*)\Delta t, \lim_{n \to \infty} \sum_{i=1}^n g(t_i*)\Delta t, \lim_{n \to \infty} \sum_{i=1}^n h(t_i*)\Delta t \right)$$

FACT

If $r(t) = (f(t), g(t), h(t)$, then

$$\int_a^b r(t)dt = \left(\int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right).$$

Theorem (Fundamental Theorem of Calculus)

Suppose that $R(t)$ and $r(t)$ are continuous vector valued functions with $R'(t) = r(t)$. Then,

$$\int_a^b r(t)dt = [R(t)]|_a^b = R(b) - R(a).$$

Note

If R and r are as above, then we will use the notation $R(t) = \int r(t)dt$ for the indefinite integral of $r(t)$.

Example

Compute $\int_0^{\pi/2}(\cos(t), \sin(t), 2t)dt$.