MTHSC 206 Section 14.3 – Arc Length and Curvature

Kevin James

Fact

Suppose that $r(t) = (x(t), y(t), z(t))$*. Then the arc length of the segment of the curve defined by* $r(t)$ *where* $a \leq t \leq b$ *is given by*

$$L = \int_a^b |r'(t)| dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt.$$

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Example

Compute the length of the arc defined by $r(t) = (\sin(\sin(t)), \cos(\sin(t)), \cos(t))$ as t varies from 0 to 2π.

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Note that the curve defined by $r(t) = (2t, t^2, \frac{1}{3}t^3)$ where $1 \leq t \leq 100$ could also be described by $q(t) = (2e^u, e^{2u}, \frac{1}{3}e^{3u})$, where $0 \leq u \leq \ln 100$.

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Example

Compute the arc length of the arc along the curve of the above example from $(2e, e^2, \frac{1}{3}e^3)$ to $(2e^2, e^4, \frac{1}{3}e^6)$.

Definition

Suppose that a curve C is parametrized by the vector function $r(t)$ as $a \leq t \leq b$ and that C is traversed exactly once as t goes from a to b. Then we define the arc length function for C as follows.

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Note

By the Fundamental Theorem of Calculus,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = |r'(t)|.$$

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Example

Let $r(t) = (\cos(t), \sin(t), t)$. Write r as a function of its arc length.

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We define the curvature of a curve by

$$\kappa = \left|\frac{dT}{ds}\right|.$$

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Compute the curvature of the circle or radius a.

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Therefore, we have $\kappa(x) = \frac{\sqrt{(f''(x))^2}}{\sqrt{1+(f'(x))^2}^3} = \frac{|f''(x)|}{(1+(f'(x))^2)^{3/2}}$.

DEFINITION

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We define the binormal vector of C at $r(t)$ as

$$B(t) = T(t) \times N(t).$$

Example

Consider the curve C parametrized by $r(t) = (\cos(t), \sin(t), 3t)$. Compute the unit tangent, the unit normal and the binormal vectors at $r(\pi) = (-1, 0, 3\pi)$.

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The plane determined by $N(t)$ and $B(t)$ is called the normal plane of C at $P = r(t)$.
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Example

Find equations of the normal and osculating planes of the curve C parametrized by $r(t) = (\cos(t), \sin(t), 3t)$ at the point $(-1, 0, 3\pi)$.

Definition

The circle that lies in the osculating plane of C at P, has the same tangent as C and lies on the concave side of C (-i.e. in the direction N points) and has radius $\rho = \frac{1}{\kappa(t)}$ is called <u>the osculating circle</u> of C at P.

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Note

Let C be a curve and S its osculating circle. Then S has the same curvature as C at P. That is, S is the circle that best indicates the behavior of the curve C near P.

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Example

Find and graph the osculating circle of the curve C parametrized by $r(t) = (\cos(t), \sin(t), t)$ at the point $(-1, 0, \pi)$.