MTHSC 206 Section 14.4 – Velocity and Acceleration

Kevin James

Definition

Suppose that an object's position at time t is given by the vector function $r(t)$. Then,

1. The velocity vector of the object at time t is given by $v(t) = r'(t)$.
2. The speed of the object is given by $|v(t)| = |r'(t) = \frac{ds}{dt}$.
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Newton's second law of motion

$$F(t) = ma(t).$$

Example

Suppose that a projectile is to be fired into the air at an angle of α from the ground with an initial velocity v_0. What is the position function for the projectile? What angle will maximize the distance the projectile will travel before returning to the ground. You many assume that the only external force acting on the projectile is gravity. (Acceleration due to gravity is 9.8 m/s^2.)

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Fact

We can decompose acceleration into its tangential and normal components as

$$a = \frac{r' \cdot r''}{|r'|} T + \frac{|r' \times r''|}{|r'|} N.$$

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FACT

Suppose that the two foci are placed at $(\pm c, 0)$ and that the constant sum of distances is $2a$. Then the points on the ellipse described above satisfy

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Fact (Alternative Definition)

Let F be a fixed point (focus) and let ℓ be a fixed line (direction) in a plane. Let e be a fixed positive number (eccentricity). The set of all points P satisfying $\frac{|PF|}{|P\ell|} = e$ is an ellipse if $e < 1$ (a parabola if $e = 1$ and a hyperbola if $e > 1$).

Fact

The polar equation for the curve described above is

$$r = \frac{ed}{1 + e\cos(\theta)},$$

where $d = |F\ell|$.

Kepler's Laws

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
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Newton's Laws of Motion and Gravitation

2nd Law of Motion $F = ma$.

Gravitation $F = \frac{-GMm}{|r|^3} r = \frac{-GMm}{|r|^2} u$.

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We may assume that $h \neq 0$, that is that r and v are not parallel.
Note that this means that $r(t) \perp h$ for all t.
So, the orbit of the planet lies in a plane with normal vector h.

Let's rewrite h as

$$\begin{aligned} h &= r \times v = r \times r' = |r| u \times (|r| u)' \\ &= |r| u \times (|r|' u + |r| u') = |r||r|'(u \times u) + |r|^2 (u \times u') \\ &= |r|^2 (u \times u'). \end{aligned}$$

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Then,

$$\begin{aligned} a \times h &= \frac{-GM}{|r|^2} u \times (|r|^2 u \times u') = -GMu \times (u \times u') \\ &= -GM[(u \cdot u')u - (u \cdot u)u'] \\ &= GMu'. \end{aligned}$$

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Thus, $(v \times h)' = a \times h = GMu'$.
Integrating both sides gives $v \times h = GMu + c$, where c is a constant vector.

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$$|r| = \frac{|h|^2 / GM}{1 + |c| / GM \cos(\theta)}.$$

Now, letting $e = \frac{|c|}{GM}$ and $d = \frac{|h|^2}{|c|}$, we see that the polar coordinates $(|r|, \theta)$, must satisfy

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which is the polar coordinates equation for a conic section.

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$$|r| = \frac{de}{1 + e\cos(\theta)},$$

which is the polar coordinates equation for a conic section. Since, the orbit of a planet is a closed curve, we deduce that $e < 1$ and that the curve is an ellipse.