MTHSC 206 Section 15.2 – Limits and Continuity

Kevin James

Example

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$x \backslash y$	-1.000	-0.667	-0.333	0.000	0.333	0.667	1.000
-1.000	0.455	0.687	0.807	0.841	0.807	0.687	0.455
-0.667	0.687	0.873	0.949	0.967	0.949	0.873	0.687
-0.333	0.807	0.949	0.992	0.998	0.992	0.949	0.807
0.000	0.841	0.967	0.998	-	0.998	0.967	0.841
0.333	0.807	0.949	0.992	0.998	0.992	0.949	0.807
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It seems reasonable to say that $\lim_{(x,y)\to(0,0)} f(x,y) = 1$.

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-0.667	0.148	0.000	0.360	1.000	0.360	0.000	0.148
-0.333	0.640	0.360	0.000	1.000	0.000	0.360	0.640
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For a better understanding of the problem try approaching $(0, 0)$ along the line $y = mx$.

Definition

Suppose that $f(x, y)$ has domain D and that (a, b) is in the interior of D. Then we say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L,$$

if for every $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and if $|(x, y), (a, b)| < \delta$, then $|f(x, y) - L| < \epsilon$.

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Note

Suppose that $f(x, y)$ is a function of two variables and that C_1 and C_2 are two curves which intersect at (a, b). If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along C_2 and if $L_1 \neq L_2$ then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

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Consider the function $f(x, y) = \frac{xy^2}{x^2+y^4}$. Does $\lim_{(x,y)\to(0,0} f(x, y)$ exit?

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4. $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}$.

Definition

We say that $f(x, y)$ is continuous at (a, b) if

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Using properties of limits, one can prove that if f and g are continuous on their domains then so are $f+g$, $f-g$, fg and f/g. Note that any zeros of g will not be in the domain of f/g.

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Fact

Polynomials and rational functions (ratios of polynomials) are continuous.

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1. Where is the function $h(x, y) = |(x, y)|$ continous?
2. Where is the function $\sin(y/x)$ continuous?

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$|\vec{x} - \vec{a}| = \sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2}$.

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Definition

Suppose that f is a real valued function defined on $D \subseteq \mathbb{R}^n$. Then we say that f is continuous at $\vec{a} \in \mathbb{R}^n$ if

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