

MTHSC 206 SECTION 15.3 – PARTIAL DERIVATIVES

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DEFINITION

Suppose that $f(x, y)$ is a function of two variables with domain D . Suppose that $(a, b) \in D$. Define $g(x) = f(x, y)$ and $h(y) = f(x, y)$. Then we define the partial derivatives of f with respect to the variables x and y as follows.

$$f_x(x, y) = g'(x), \quad f_y(x, y) = h'(y).$$

FACT

$$f_x(a, b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h} \quad \text{and}$$
$$f_y(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b+h) - f(a, b)}{h}.$$

NOTATION

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f,$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f.$$

EXAMPLE

Compute the partial derivatives with respect to x and y for the following functions.

① $f(x, y) = x^4 + y^3 + 3xy.$

② $g(x, y) = \sin(xy).$

NOTE

Suppose that $f(x, y)$ is a function with domain D and that $(a, b) \in D$. Consider the curves $C_x : z = f(x, b)$ and $C_y : z = f(a, y)$. The slope of the line in the plane $y = b$ which is tangent to C_x at $(a, b, f(a, b))$ is $f_x(a, b)$.

The slope of the line in the plane $x = a$ which is tangent to C_y at $(a, b, f(a, b))$ is $f_y(a, b)$.

The equations of the tangent lines are

$$\begin{cases} z - c = f_x(a, b)(x - a) \\ y = b. \end{cases} \quad \begin{cases} z - c = f_y(a, b)(y - b) \\ x = a. \end{cases}$$

They can be parametrized by the vector functions

$$\begin{aligned} r_x(t) &= (a, b, c) + t(1, 0, f_x(a, b)) \quad \text{and} \\ r_y(t) &= (a, b, c) + t(0, 1, f_y(a, b)). \end{aligned}$$

EXAMPLE

- 1 Find the equations of the tangent lines to the graph of $f(x, y) = \sin(xy)$ at the point $(\pi, \frac{1}{2}, 1)$.
- 2 Find the equations of the tangent lines to the sphere $x^2 + y^2 + z^2 = 9$ at the point $(\sqrt{3}, \sqrt{3}, \sqrt{3})$.

NOTE

We can of course consider the partial derivatives of functions of many variables. Suppose that $f(\vec{x})$ is a function of the n variables x_1, \dots, x_n . Then we define

$$f_{x_i}(\vec{a}) = \lim_{h \rightarrow 0} \frac{f(a_1, \dots, a_{i-1}, a_i + h, a_{i+1}, \dots, a_n) - f(a_1, \dots, a_n)}{h}.$$

We can of course use the rules of differentiating functions of one variable to compute these partials.

EXAMPLE

Compute the partial derivatives of $f(x, y, z) = x \sin(y)e^z$.

NOTE

We can of course consider higher derivatives. The notation for the second order derivatives is as follows.

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2},$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x},$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y},$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2}.$$

The notation for third and higher order derivatives is similar.

EXAMPLE

Let $f(x, y) = x^2 - xy + y^2$. Compute the 2nd order partial derivatives of f .

THEOREM (CLAIRAUT)

Suppose that f is defined at (a, b) . If there is a disk D containing (a, b) on which f_{xy} and f_{yx} are continuous, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$