# MTHSC 206 Section 15.4 – Tangent Planes and Linear Approximations

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## Fact

*Suppose that $f$ has continuous partial derivatives at $(x_0, y_0)$. Then the tangent plane of $f$ at $(x_0, y_0)$ (i.e. the plane determined by the tangent lines to $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$. has the equation*

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

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### Example

Find the equation of the plane tangent to $z = 3x^2 + 5y^2$ at the point $(1, 1, 8)$.

**Definition**

Suppose that $f$ has continuous partial derivatives. We define the linearization of $f$ at $(x_0, y_0)$ as

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0).$$

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**Example**

Approximate the value of $3x^2 + 5y^2$ at $(1.01, 1.01)$.

### Definition

The function $z = f(x, y)$ is differentiable at $(a, b)$ if the quantity $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$ can be expressed as

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

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## Theorem

*If $f_x$ and $f_y$ are defined and continuous near $(a, b)$, then $f$ is differentiable at $(a, b)$.*

### Example

Show that $f(x, y) = y\sin(xy)$ is differentiable at $(\frac{1}{2}, \pi)$. Find its linearization at $(\frac{1}{2}, \pi)$ and use it to approximate $f(.55, 3)$.

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Given a function $f(x, y)$, we define differentials $dx$ and $dy$ to be independent variables. We define the total differential dz as

$$\mathrm{dz} = f_x(x, y)\mathrm{dx} + f_y(x, y)\mathrm{dy}.$$

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In the above expression, the dependent variable dz depends on the 4 independent variables $x, y$, dx and dy.

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### Note

If $f$ has continuous partials, then $f(x + \mathrm{dx}, y + \mathrm{dy}) \approx f(x,y) + \mathrm{dz}$.

## Example

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$$dz = \sum_{i=1}^{n} f_{x_i}(\vec{x}) dx_i.$$

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*If $f$ is differentiable, then it will be true that*

$$f(\vec{x} + \vec{dx}) \approx f(\vec{x}) + dz.$$