MTHSC 206 Section 15.6 – Directional Derivatives and the Gradient Vector

Kevin James

Definition

We define the directional derivative of the function $f(x, y)$ at the point (x_0, y_0) in the direction of the unit vector $u = (a, b)$ (u should be thought of as a vector in the xy-plane) as

$$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + by) - f(x_0, y_0)}{h}.$$

Theorem

If $f(x, y)$ is differentiable, then

$$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

Proof.

Define $g(h) = f(x_0 + ah, y_0 + bh)$.
Then, we have that

$$\begin{aligned} g'(0) &= \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h} \\ &= D_u f(x_0, y_0). \end{aligned}$$

We can also write $g(h) = f(x, y)$ where $x = x_0 + ah$ and $y = y_0 + bh$.
Applying the chain rule, we have,

$$g'(h) = f_x(x, y)a + f_y(x, y)b.$$

Thus,

$$D_u f(x_0, y_0) = g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b.$$

Note

If the vector u is at an angle θ with the x-axis then we can write $u = (\cos(\theta), \sin(\theta))$. Thus

$$D_u f(x, y) = f_x(x, y)\cos(\theta) + f_y(x, y)\sin(\theta).$$

Example

Find the directional derivative $D_u f(x, y)$ of the function $f(x, y) = x^2 + xy + y^2$ in the direction of the unit vector which is at an angle of $\theta = \frac{\pi}{3}$ to the x-axis.

Note

The directional derivative of f in the direction of u can be written as

$$D_u f(x, y) = (f_x(x, y), f_y(x, y)) \cdot u.$$

Definition

We define the gradient of a function $f(x, y)$ as

$$\nabla f = (f_x(x, y), f_y(x, y)) = f_x(x, y)i + f_y(x, y)j.$$

Fact

If u is a unit vector and $f(x, y)$ is a function of 2 variables then

$$D_u f(x, y) = \nabla f \cdot u.$$

Example

Consider the function $f(x, y) = e^{xy}$. Compute the gradient of f. Compute the directional derivative of f in the direction of $u = (\sqrt{3}/2, 1/2)$.

Definition

The <u>directional derivative</u> of $f(x, y, z)$ at (x_0, y_0, z_0) in the direction of the unit vector $u = (a, b, c)$ is

$$D_u f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh, z_0 + ch) - f(x_0, y_0, z_0)}{h}.$$

if this limit exists.

Fact

If $f(x, y, z)$ is differentiable then

$$D_u f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c.$$

Definition

We define the gradient of $f(x, y, z)$ as

$$\nabla f = (f_x, f_y, f_z).$$

Fact

$$D_u f(x, y, z) = \nabla f(x, y, z) \cdot u$$

Example

Suppose that $f(x, y, z) = sin(xy)e^z$. Compute ∇f. What is the directional derivative at $(\pi, 1/2, 0)$ in the direction $(\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3)$. Can you find the direction which maximizes $D_u f$ at this point?

Theorem

Suppose that f is a differentiable function of two or three variables. The maximal value of the directional derivative $D_u f(\vec{x})$ at the point $\vec{x}$ is $|\nabla f|$ and it occurs when $u = \frac{1}{|\nabla f|}\nabla f$.

Example

Consider the function $f(x, y, z) = e^{xyz}$. What is the directional derivative at the point $(0, 1, 0)$ in the direction of $\overrightarrow{((0, 1, 0), (1, 1, 1))}$. What is the maximum value of the directional derivative at this point? In which direction does it occur?

Suppose that S is the level surface of $F(x, y, z)$ given by $F(x, y, z) = k$ and $P = (x_0, y_0, z_0)$ is a point on S.
Let C be any curve that lies on S and passes through P.
Then C can be parametrized by $r(t) = (x(t), y(t), z(t))$.
Suppose that $r(t_0) = P$.
Note that $F(x(t), y(t), z(t)) = k$ because C lies on S.
Supposing all functions to be differentiable, we can use the chain rule to obtian,

$$0 = F_x x'(t) + F_y y'(t) + F_z z'(t) = \nabla F \cdot r'(t).$$

That is, $\nabla F(P)$ is orthogonal to the tangent vector at P of any curve along S passing through P.

Definition

We define the <u>tangent plane to the level surface $F(x, y, z) = k$ at $P = (x_0, y_0, z_0)$</u> as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. This plane has equation

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

or

$$\nabla F(x_0, y_0, z_0) \cdot \overrightarrow{(P, (x, y, z))} = 0$$

Definition

We define the normal line to the level surface $F(x, y, z) = k$ at P to be the line passing through P and orthogonal to the tangent plane, that is the line through P parallel to the normal vector for the tangent plane which is $\nabla F(P)$. This line has symmetric equations

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}.$$

Note

We can think of the graph $z = f(x, y)$ of $f(x, y)$ as the level surface $F(x, y, z) = 0$ where $F(x, y, z) = f(x, y) - z$.
In this case, $\nabla F = (f_x, f_y, -1)$
So the tangent plane to the graph of f as a level surface at P would have equation

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) - (z - z_0) = 0.$$

which is consistent with our previous definition of tangent plane.
The normal line has equation

$$\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = -(z - z_0).$$

Example

Find the equations of the tangent plane and normal line at the point $(1,1,2)$ to the ellipsoid $\frac{x^2}{9}+\frac{y^2}{4}+z^2=5$.