MTHSC 206 Section 15.7 – Maximum and Minimum Values

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Definition

- We say that a function $f(x, y)$ has a local maximum at (a, b) if there is a disk D centered at (a, b) for which $f(x, y) \leq f(a, b)$ for all (x, y) in D. In this case, $f(a, b)$ is called a local maximum value of f.
- We say that a function $f(x, y)$ has a local minimum at (a, b) if there is a disk D centered at (a, b) for which $f(x, y) \geq f(a, b)$ for all (x, y) in D. In this case, $f(a, b)$ is called a local minimum value of f.
- Suppose that $f(x, y)$ has domain $\mathcal{D}$. We say that $f(x, y)$ has an absolute maximum at (a, b) if $f(x, y) \leq f(a, b)$ for all (x, y) in $\mathcal{D}$. In this case, $f(a, b)$ is called an absolute maximum value of f.
- We say that $f(x, y)$ has an absolute minimum at (a, b) if $f(x, y) \geq f(a, b)$ for all (x, y) in $\mathcal{D}$. In this case, $f(a, b)$ is called an absolute minimum value of f.

Theorem

Suppose that $f(x, y)$ has a local extreme at (a, b) and that f_x and f_y exist at (a, b). Then $f_x(a, b) = f_y(a, b) = 0$.

Definition

Suppose that $f(x, y)$ has domain $\mathcal{D}$ and that $(a, b) \in \mathcal{D}$. We say that (a, b) is a critical point if $f_x(a, b) = f_y(a, b) = 0$ or if one of f_x or f_y does not exist at (a, b).

Note

From our theorem above, we see that local extrema occur at critical points. However, it is not necessarily true that all critical points are locations of local extrema.

Example

Consider $f(x, y) = x^2 + y^2 - x2 + 6y + 10$. Identify the critical points and discuss the possibility of local extrema.

Example (Beware Saddle Points)

Consider $f(x, y) = y^2 - x^2$. Identify the critical points and discuss the possibility of local extrema.

Theorem (2nd Derivative Test)

Suppose that the second order partials of $f(x, y)$ exist and are continuous on a disk centered at (a, b). Further suppose that $f_x(a, b) = f_y(a, b) = 0$. Let

$$D = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

1. *If $D > 0$ and $f_{xx}(a, b) > 0$ then (a, b) is a local minimum.*
2. *If $D > 0$ and $f_{xx}(a, b) < 0$ then (a, b) is a local maximum.*
3. *If $D < 0$ then (a, b) is not a local extreme.*

Note

1. In case 3, (a, b) is called a saddle point and the graph of $f(x, y)$ crosses its tangent plane.
2. If $D = 0$ then we have no information, (a, b) could be a local max or local min or a saddle point.
3. Note that $D = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$.

Example

Find the maximum and minimum values of the function $f(x, y) = x^4 + y^4 - 4xy + 1$.

Example

Suppose that we wish to construct a box with no lid using $10m^2$ of cardboard in such a way as to maximize the volume of the box. What dimensions should we choose?

Definition

- A boundary point of a set D in $\mathbb{R}^2$ is a point (a, b) such that every disk centered at (a, b) contains points in D and points not in D.
- A closed set in $\mathbb{R}^2$ is a set that contains all of its boundary points.
- A bounded set in $\mathbb{R}^2$ is a set which is completely contained in some disk.
- A set in $\mathbb{R}^2$ which is both closed and bounded is called compact.

Theorem (Extreme Values)

If f is continuous on a compact set D in $\mathbb{R}^2$, then f attains absolute maximum and absolute minimum values on D.

Finding absolute Extrema on a Compact Set

We use the following procedure to find the absolute extreme values of a continuous function f on a compact set D.

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from the previous steps is the absolute maximum value of f on D. The smallest of the values from the previous steps is the absolute minimum value of f on D.

Example

Find the extreme values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leq x \leq 4, -1 \leq y \leq 3\}$.