# MTHSC 206 Section 16.1 – Double integrals over rectangles

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## Idea

Suppose that we want to compute the volume underneath the graph of a positive valued function $f(x, y)$ as $x$ and $y$ vary over the rectangle $R = [a, b] \times [c, d]$.
Motivated by our success with functions of one variable, we subdivide $R$ into small rectangles.
Set $\Delta x = \frac{b-a}{n}$ and $\Delta y = \frac{d-c}{m}$ for some integers $m$ and $n$.
Let $x_i = a + i\Delta x$ and $y_j = c + j\Delta y$.
The volume underneath $f(x, y)$ and directly above the rectangle $R_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ can be approximated by

$$f(x_{ij}, y_{ij})\Delta x\Delta y = f(x_{ij}, y_{ij})\Delta A,$$

where $(x_{ij}, y_{ij})$ is a point lying in $R_{ij}$. Thus we approximate the volume underneath $R$ by

$$V \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}, y_{ij})\Delta A.$$

## Fact

*If the function $f(x, y)$ is continuous then the volume lying below the graph of $f(x, y)$ and directly above the rectangle $R = [a, b] \times [c, d]$ is given by*

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}, y_{ij})\Delta A.$$

## Definition

We define the <u>double integral</u> of $f(x, y)$ over the rectangle $R = [a, b] \times [c, d]$ by

$$\int\int_R f(x, y)\ \mathrm{dA} = \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{ij}, y_{ij})\Delta A$$

## Definition

A function $f$ is called integrable if the limit in the previous definition exists.

## Note

We could choose our point $(x_{ij}, y_{ij})$ to be the corner point $(x_i, y_j)$. Then the expression for the double integral becomes

$$\int\int_R f(x,y)\ \mathrm{dA} = \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\Delta A$$

### Example

Estimate the area underneath the graph of $f(x, y) = xy$ and directly above the rectangle $[1, 5] \times [1, 5]$. Can you compute the area?

### Example

Let $R = [-1, 1] \times [-3, 3]$. Interpret the integral $\int\int_R \sqrt{1 - x^2}\, dA$ as a volume and calculate its value exactly.

## Midpoint Rule for Double Integrals

$$\int\int_R f(x,y)\,\mathrm{dA} \approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(\bar{x}_i,\bar{y}_j)\Delta A,$$

where $\bar{x}_i = x_{i-1} + \frac{\Delta x}{2}$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_i = y_{i-1} + \frac{\Delta y}{2}$ is the midpoint of $[y_{i-1}, y_i]$.

## Example

Let $R = [0,4] \times [0,4]$. Use the midpoint rule to estimate $\int\int_R (x^2 + y^2)\,\mathrm{dA}$.

## Note

We can estimate the average value of the function $f(x, y)$ over the rectangle $R = [a, b] \times [c, d]$ by

$$\begin{aligned} \frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j) &= \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j)\frac{\Delta A}{A} \\ &= \frac{1}{A}\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j)\Delta A. \end{aligned}$$

### Definition

We define the average value of the function $f(x, y)$ over the rectangle $R = [a, b] \times [c, d]$ by

$$f_{\text{ave}} = \frac{1}{A(R)} \int \int_R f(x, y) \; \mathrm{dA}.$$

### Example

Estimate the average value of the function $f(x, y) = xy$ on the rectangle $R = [1, 5] \times [1, 5]$.

## Properties of Double Integrals

1. $\int\int_R [f(x,y)+g(x,y)]\ \mathrm{dA} = \int\int_R f(x,y)\ \mathrm{dA} + \int\int_R g(x,y)\ \mathrm{dA}$.
2. If $c \in \mathbb{R}$, $\int\int_R cf(x,y)\ \mathrm{dA} = c\int\int_R f(x,y)\ \mathrm{dA}$.
3. If $f(x,y) \geq g(x,y)$ for all points $(x,y)$ in $R$ then $\int\int_R f(x,y)\ \mathrm{dA} \geq \int\int g(x,y)\ \mathrm{dA}$.