MTHSC 206 Section 16.2 – Iterated Integrals

Kevin James

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Definition

Given a function $f(x, y)$ which is integrable over a rectangle $R = [a, b] \times [c, d]$, the integrals

$$\int_a^b \left[\int_c^d f(x, y)\ \mathrm{dy} \right]\ \mathrm{dx} \quad \text{and} \quad \int_c^d \left[\int_a^b f(x, y)\ \mathrm{dx} \right]\ \mathrm{dy}$$

are called iterated integrals.

Example

Compute the two iterated integrals $\int_0^2 \left[\int_1^3 xy \ \mathrm{dy}\right] \ \mathrm{dx}$ and $\int_1^3 \left[\int_0^2 xy \ \mathrm{dx}\right] \ \mathrm{dy}$

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Example

Let $R = [0,2] \times [1,3]$. Compute $\int\int_R xy \text{ dA}$.

Theorem (Fubini)

If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\begin{aligned} \int\int_R f(x, y)\ dA &= \int_a^b \left[\int_c^d f(x, y)\ dy\right]\ dx \\ &= \int_c^d \left[\int_a^b f(x, y)\ dx\right]\ dy \end{aligned}$$

More generally the result holds if f is bounded on R, f is discontinuous only on a finite number of smooth curves and the iterated integrals exits.

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Example

Let $R = [1,2] \times [2,3]$. Compute $\int\int_R (x^2 + y)\, dA$.

Corollary

Suppose that $f(x,y) = g(x)h(y)$ satisfies the hypotheses of Fubini's theorem on the rectangle $R = [a,b] \times [c,d]$. Then,

$$\int\int_R f(x,y)\, dA = \left(\int_a^b g(x)\, dx\right) \cdot \left(\int_c^d h(y)\, dy\right).$$

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Example

Let $R = [0,2] \times [1,3]$. Compute $\int\int_R xy$ dA.

Example (Order of integration matters)

Suppose that $R = [0, \pi] \times [1, 2]$. Compute $\int\int_R x\cos(xy)\ dA$ using both iterated integral formulas.