# MTHSC 206 Section 16.4 – Double Integrals in Polar Coordinates

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## Recall

The relationship between the Euclidean coordinates $(x, y)$ of a point $P$ and the polar coordinates $(r, \theta)$ of the same point $P$ are given by

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### Goal

We will revisit our development of the double integral in terms of polar coordinates.

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The volume underneath $f(x, y)$ and directly above the rectangle $R_{ij} = \{(r, \theta) \mid r_{i-1} \leq r \leq r_i; \theta_{j-1} \leq \theta \leq \theta_j\}$ can be approximated by

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where $\bar{r}_i = \frac{r_i+r_{i-1}}{2}$, $\bar{\theta}_j = \frac{\theta_j-\theta_{j-1}}{2}$ and $\Delta A_{ij}$ is the area of $R_{ij}$.

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Thus we approximate the volume underneath $R$ by

$$V \approx \sum_{i=1}^{n} \sum_{j=1}^{m} f(\bar{r}_i \cos(\bar{\theta}_j), \bar{r}_i \sin(\bar{\theta}_j))\bar{r}_i \Delta r \Delta\theta.$$

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## THEOREM

*If $f(x, y)$ is continuous on a polar rectangle $R$ as above, then*

$$\int\int_R f(x, y)\ dA = \int_a^b \int_\alpha^\beta rf(r\cos(\theta), r\sin(\theta))\ dr\ d\theta$$

### Example

Evaluate $\int\int_R (3x + 4y^2)$ dA where $R$ is the region bounded by the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

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Compute the volume of a silo of hight 6 meters and diameter 2 meters whose top is a semi-sphere of radius 1 meter.

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*Suppose that $f$ is continuous on*

$$D = \{(r, \theta) \mid h_1(\theta) \leq r \leq h_2(\theta); \alpha \leq \theta \leq \beta\}.$$

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$$\int\int_D f(x, y)\ dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} rf(r\cos(\theta), r\sin(\theta)\ dr\ d\theta.$$

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*In particular,*

$$A(D) = \int\int_D 1\ dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} r\ dr\ d\theta = \int_\alpha^\beta \frac{h_2(\theta)^2 - h_1(\theta)^2}{2}\ d\theta.$$

### Example

Compute the area of one petal of the five-petaled flower like region $D = \{(r, \theta) \mid 0 \leq r \leq \cos(5\theta); \frac{-\pi}{10} \leq \theta \leq \frac{\pi}{10}\}$.

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### Example

Compute the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, inside the cylinder $x^2 + y^2 = 2x$ and above the $xy$-plane.