MTHSC 206 Section 16.5 – Applications of Double Integrals

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Mass and Density

Suppose that a lamina represented by a region D of $\mathbb{R}^2$ has variable density given by $\rho(x, y)$. Then the mass of the lamina can be computed by

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Electrical Charge

If an electrical charge is distributed over a region D of $\mathbb{R}^2$, and the charge density (in units of charge per unit area) is given by $\sigma(x, y)$, then the total charge is given by

$$Q = \int \int \sigma(x, y) \; \mathrm{dA}.$$

Example

Suppose that a charge is distributed over the triangular region $D = \{(x, y) \mid 0 \leq x \leq 1; 1 - x \leq y \leq 1\}$ with charge density $\sigma(x, y) = xy \ C/m^2$. Find the total charge over the region D.

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Thus the total moment of the lamina with respect to the x-axis is given by

$$M_x = \int\int_D y\rho(x, y)\ \mathrm{dA}.$$

Similarly, the total moment of the lamina with respect to the y-axis is given by

$$M_y = \int\int_D x\rho(x, y)\ \mathrm{dA}.$$

Definition

We define the center of mass (or center of gravity) of a lamina occupying a region D to be the point $(\bar{x}, \bar{y})$ such that $m\bar{x} = M_y$ and $m\bar{y} = M_x$, where m is the total mass of the lamina.

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Fact

The center of mass $(\bar{x}, \bar{y})$ of a lamina occupying a region D of $\mathbb{R}^2$ is given by

$$\bar{x} = \frac{M_y}{m} = \frac{\int\int_D x\rho(x,y)\ dA}{\int\int_D \rho(x,y)\ dA}, \quad \bar{y} = \frac{M_x}{m} = \frac{\int\int_D y\rho(x,y)\ dA}{\int\int_D \rho(x,y)\ dA}.$$

Example

Suppose that the density at a point (x, y) of a semicircular lamina is proportional to the distance from (x, y) to the center of the circle. Find the center of mass for the lamina.

Definition

Suppose that X and Y are random variables. The joint density function of X and Y is a function $f(X, Y)$ such that $P\left[(X, Y) \in D\right] = \int\int_D f(X, Y)\, \mathrm{d}A$.

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Example

Suppose that the joint density function for X and Y is given by

$$f(X, Y) = \begin{cases} C(x+y) & \text{if } 0 \leq y \leq 10, \text{ and } 0 \leq x \leq 10, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the constant C. Then calculate $P[X \leq 7; Y \geq 2]$.