MTHSC 206 Section 16.6 – Triple Integrals

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Definition

The triple integral of $f(x, y, z)$ over the box $B = \{(x, y, z) \mid a \leq x \leq b; c \leq y \leq d; r \leq z \leq s\}$ is defined by

$$\int\int\int_B f(x, y, z)\ \mathrm{dV} = \lim_{l,m,n \to 0} \sum_{i=0}^{l} \sum_{j=0}^{m} \sum_{k=0}^{n} f(x_i, y_j, z_k) \Delta V.$$

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Theorem (Fubini)

If $f(x, y, z)$ is continuous on B then,

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Note

There are a total of six possible orders of integration for triple integrals. A more general version of Fubini's theorem guarantees that these all have the same value.

Example

Evaluate the triple integral $\int\int\int_B xyz \, \mathrm{dV}$ where $B = [0,1] \times [0,1] \times [0,1]$.

Definition

We consider three types of regions is $\mathbb{R}^3$.

TYPE 1 $E = \{(x, y, z) \mid (x, y) \in D; u_1(x, y) \le z \le u_2(x, y)\}$

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Fact

If E is a type 1,2 or 3 region, then $\int\int\int_E f(x,y,z)\, dV$ has the same value as

$$\begin{cases} \int\int_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\right] dA & \text{if } E \text{ is type 1,} \\ \int\int_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\, dx\right] dA & \text{if } E \text{ is type 2,} \\ \int\int_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\, dy\right] dA & \text{if } E \text{ is type 3.} \end{cases}$$

Example

Evaluate $\int\int\int_E z \text{ dV}$ where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

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Evaluate $\int\int\int_E \sqrt{y^2 + z^2} \text{ dV}$, where E is the solid bounded by the paraboloid $x = y^2 + z^2$ and the plane $x = 4$.

Volume

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Mass and density

Suppose that E is a solid in $\mathbb{R}^3$ with density function $\rho(x, y, z)$. Then the mass of E is given by $m = \int\int\int \rho(x, y, z) \, \mathrm{dV}$.

Moments and Center of Mass

Suppose that E is a solid in $\mathbb{R}^3$ with density function $\rho(x,y,z)$. The moments of E about the three coordinate planes are

$$\begin{aligned} M_{yz} &= \int\int\int_E x\rho(x,y,z)\ \mathrm{dV}, \quad M_{xz} = \int\int\int_E y\rho(x,y,z)\ \mathrm{dV}, \\ M_{xy} &= \int\int\int_E z\rho(x,y,z)\ \mathrm{dV}. \end{aligned}$$

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Thus the center of mass of E is at $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m} \quad \text{and} \quad \bar{z} = \frac{M_{xy}}{m}$$

Example

Find the center of mass of the solid of constant density ρ that is bounded by the parabolic cylinder $y = x^2$ and the planes $z = 0$ and $z = 1$.