MTHSC 206 Section 16.7 – Triple Integrals in Cylindrical Coordinates

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Fact

We can describe any point $P = (x, y, z)$ in $\mathbb{R}^3$ by specifying the projection (x, y) of P onto the xy-plane in polar coordinates (r, θ) and specifying the height z of P.
The triple (r, θ, z) is called the cylindrical coordinates of P.
The relationship between the Euclidean coordinates (x, y, z) and the cylindrical coordinates (r, θ, z) is given by

$$
\begin{aligned}
x &= r\cos(\theta), \quad y = r\sin(\theta), \quad z = z. \\
r^2 &= x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z.
\end{aligned}
$$

Example

1. The point with cylindrical coordinates $(1, \frac{2\pi}{3}, 3)$ has Euclidean coordinates $(\frac{1}{2}, \frac{\sqrt{3}}{2}, 3)$.
2. The point with Euclidean coordinates $(2, 2, -7)$ had cylindrical coordinates $(2\sqrt{2}, \frac{\pi}{4}, -7)$.

Example

Describe the surface whose cylindrical coordinates satisfy $z = r$.

Fact

Suppose that $E = \{(x,y,z) \mid (x,y) \in D; u_1(x,y) \leq z \leq u_2(x,y)\}$ *where* $D = \{(r,\theta) \mid \alpha \leq \theta \leq \beta; h_1(\theta) \leq r \leq h_2(\theta)\}$. *Then*

$$\int\int\int_E f(x,y,z)\ dV = \int\int_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz\right]\ dA$$

$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \left[\int_{u_1(r\cos(\theta),r\sin(\theta))}^{u_2(r\cos(\theta),r\sin(\theta))} f(r\cos(\theta), r\sin(\theta), z)\ dz\right] r\ dr\ d\theta.$$

Example

Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}\left(x^2+y^2\right)\ dz\ dy\ dx$.