

MTHSC 206 SECTION 16.7 – TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES

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FACT

We can describe any point $P = (x, y, z)$ in $\mathbb{R}^3$ by specifying the projection (x, y) of P onto the xy -plane in polar coordinates (r, θ) and specifying the height z of P .

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$$\begin{aligned}x &= r \cos(\theta), & y &= r \sin(\theta), & z &= z. \\r^2 &= x^2 + y^2, & \tan(\theta) &= \frac{y}{x}, & z &= z.\end{aligned}$$

EXAMPLE

- 1 The point with cylindrical coordinates $(1, \frac{2\pi}{3}, 3)$ has Euclidean coordinates $(\frac{1}{2}, \frac{\sqrt{3}}{2}, 3)$.

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EXAMPLE

Describe the surface whose cylindrical coordinates satisfy $z = r$.

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Suppose that $E = \{(x, y, z) \mid (x, y) \in D; u_1(x, y) \leq z \leq u_2(x, y)\}$ where $D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta; h_1(\theta) \leq r \leq h_2(\theta)\}$. Then

$$\int \int \int_E f(x, y, z) \, dV = \int \int_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \right] \, dA$$

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EXAMPLE

Evaluate $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2 + y^2) \, dz \, dy \, dx$.