MTHSC 206 Section 16.8 – Triple Integrals in Spherical Coordinates

Kevin James

Definition

We can express the location any point $P = (x, y, z)$ in $\mathbb{R}^3$ by specifying

1. the angle θ between the x-axis and the line from O to the projection of P into the xy-plane $(x, y, 0)$ and
2. the angle ϕ between the z-axis and the line segment $\overline{OP}$.
3. the distance ρ from P to the origin O,

The triple (ρ, θ, ϕ) is called the spherical coordinates of P.

Note

We note that $\rho \geq 0$, $0 \leq \theta < 2\pi$ and $0 \leq \phi \leq \pi$.

Fact

The relationship between the spherical coordinates (ρ, θ, ϕ) and the Euclidean coordinates (x, y, z) is given by

$$x = \rho\sin(\phi)\cos(\theta), \quad y = \rho\sin(\phi)\sin(\theta), \quad z = \rho\cos(\phi)$$
$$\rho^2 = x^2 + y^2 + z^2$$

Example

1. The Euclidean coordinates of the point with spherical coordinates $(3, \frac{\pi}{3}, \frac{\pi}{4})$ are $(\frac{3\sqrt{6}}{4}, \frac{3\sqrt{2}}{4}, \frac{3\sqrt{2}}{2})$.
2. The spherical coordinates of the point with Euclidean coordinates $(6, 6, 2\sqrt{6})$ are $(4\sqrt{6}, \frac{\pi}{4}, \frac{2\pi}{3})$.

Note

The spherical wedge determined by a change of angles $\Delta\theta$ from θ and $\Delta\phi$ from ϕ and a change of radius $\Delta\rho$ from ρ is $\rho^2 \sin(\phi)\Delta\rho\Delta\theta\Delta\phi$.

Thus if E is the spherical wedge $\{(\rho, \theta, \phi) \mid a \leq \rho \leq b; \alpha \leq \theta \leq \beta; c \leq \phi \leq d\}$, then we can derive the following formula for triple integration in spherical coordinates.

$$\int\int\int_E f(x, y, z)\, \mathrm{d}V =$$

$$\int_c^d \int_\alpha^\beta \int_a^b \rho^2 \sin(\phi) f(\rho \sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\, \mathrm{d}\rho\, \mathrm{d}\theta\, \mathrm{d}\phi$$

Example

Evaluate $\int\int\int_B e^{(x^2+y^2+z^2)^{3/2}}$ dV where B is the unit ball about the origin.

Example

Find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$.