MTHSC 206 Section 16.9 – Change of Variables in Multiple Integrals

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Recall

In one variable calculus we recall the change of variable formula for integration is

$$\int_a^b f(x)\,\mathrm{dx} = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)\,\mathrm{du}$$

where we have substituted $x = g(u)$. We are assuming that g is one to one on $[a, b]$ and that g is continuous.

Heuristic Explanation

Suppose that we take $\Delta u = \frac{g^{-1}(b) - g^{-1}(a)}{n}$, $u_i = g^{-1}(a) + i\Delta u$.
Then take $x_i = g(u_i)$ so that $\Delta x = g(u_i) - g(u_{i-1})$.
Note that $\Delta x = g(u_{i-1} + \Delta u) - g(u_{i-1}) \approx g'(u_{i-1})\Delta u$.

$$\begin{aligned}
\int_a^b f(x)\,\mathrm{dx} &\approx \sum_{i=1}^{n} f(x_{i-1})\Delta x \\
&\approx \sum_{i=1}^{n} f(g(u_{i-1}))(g'(u_{i-1})\Delta u) \\
&\approx \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)\,\mathrm{du}.
\end{aligned}$$

Note

In the one variable change of variables formula, we replace dx with $g'(u)$ du because when $x = g(u)$ and g is differentiable, $\Delta x \approx g'(u)\Delta u$. That is, our measure of length changes when we replace the interval $[a, b]$ with the interval $[g(a), g(b)]$.

Two Variable Integration

Suppose now that we wish to integrate $f(x, y)$ over R.
Suppose that we have a differentiable 1-1 function $T(u, v) = [g(u, v), h(u, v)]$ with $T(S) = R$.
We would like to replace $\int\int_R f(x, y)\, dA$ with an integral over the region S.
Now, we can proceed as before. Subdivide S into rectangles $S_{ij} = [u_{i-1}, u_i] \times [v_{i-1}, v_i]$ with dimensions Δu and Δv as usual.
Subdivide R into subregions $R_{ij} = T(S_{ij})$.

Two Variable Integration continued

Then,

$$\int\int_R f(x,y)\,dA \approx \sum_{i,j} f(x_{i-1},y_{j-1})\text{Area}(R_{ij})$$

$$\approx \sum_{i,j} f(g(u_{i-1},v_{j-1}),h(u_{i-1},v_{j-1}))\text{Area}(T(S_{ij}))$$

$$\approx \sum_{i,j} f(g(u_{i-1},v_{j-1}),h(u_{i-1},v_{j-1}))\left|\overrightarrow{\Delta T_{u,i-1,j-1}} \times \overrightarrow{\Delta T_{v,i-1,j-1}}\right|,$$

where

$\overrightarrow{\Delta T_{u,i-1,j-1}} = T(u_{i-1}+\Delta u, v_{j-1}) - T(u_{i-1},v_{j-1}) \approx \Delta u \overrightarrow{T_u(u_{i-1},v_{j-1})}$

and

$\overrightarrow{\Delta T_{v,i-1,j-1}} = T(u_{i-1}, v_{j-1}+\Delta v) - T(u_{i-1},v_{j-1}) \approx \Delta v \overrightarrow{T_v(u_{i-1},v_{j-1})}$.

Thus, $\int\int_R f(x,y)\,dA$ is approximated by

$$\sum_{i,j} f(g(u_{i-1},v_{j-1}),h(u_{i-1},v_{j-1}))\left|\overrightarrow{T_u(u_{i-1},v_{j-1})} \times \overrightarrow{T_v(u_{i-1},v_{j-1})}\right|\Delta u \Delta v.$$

Note

Since $T = [g, h]$, we have
$\left|\overrightarrow{T_u} \times \overrightarrow{T_v}\right| = \left|\det\begin{pmatrix} i & j & k \\ g_u & h_u & 0 \\ g_v & h_v & 0 \end{pmatrix}\right| = \left|\det\begin{pmatrix} g_u & h_u \\ g_v & h_v \end{pmatrix} k\right| =$
$\left|\det\begin{pmatrix} g_u & g_v \\ h_u & h_v \end{pmatrix}\right|.$

Definition

We define the Jacobian of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ by

$$\frac{\partial(x, y)}{\partial(u, v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Theorem

Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type 1 or type 2 plane regions. Suppose that T is one-to-one except perhaps on the boundary of S. Then

$$\int\int_R f(x,y)\,dA = \int\int_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv.$$