MTHSC 206 Section 17.2 – Line Integrals

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Line Integrals

We wish to develop the notion of integrating a real valued function $f(x, y)$ along a smooth curve.
Suppose that C is a smooth curve which is parametrized by $r(t) = [x(t), y(t)]$ as $a \leq t \leq b$.
For $n \geq 1$, we define $\Delta t = \frac{b-a}{n}$, $t_i = a + i\Delta t$ and $s_i = r(t_i) = [x(t_i), y(t_i)] = [x_i, y_i]$.
Also, we define Δs_i to be the length of the arc form $r(t_{i-1})$ to $r(t_i)$.

Definition

We define the line integral of f along C by

$$\int_C f(x, y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1}, y_{i-1}) \Delta s_i$$

Note

We note that $\Delta s_i \approx |s_i - s_{i-1}| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$. Since $x_i - x_{i-1} = x(t_{i-1} + \Delta t) - x(t_{i-1}) \approx x'(t_{i-1})\Delta t$ and similarly for $y_i - y_{i-1}$, $\Delta s_i \approx \left(\sqrt{x'(t_{i-1})^2 + y'(t_{i-1})^2}\right)\Delta t$.

Thus, one can prove

$$\int_C f(x,y)\ \mathrm{ds} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1}, y_{i-1}) \left(\sqrt{x'(t_{i-1})^2 + y'(t_{i-1})^2}\right) \Delta t$$

Fact

$\int_C f(x,y)\ ds = \int_a^b f(x(t), y(t)) \sqrt{(x'(t))^2 + (y'(t))^2}\ dt.$

Example

Evaluate $\int_C (2 + x^2 y)\, ds$ where C is the upper half of the unit circle.

Example

Evaluate $\int_C (x^2 + y^2)\, ds$ along the straight line from $(a, 0)$ to $(b, 0)$.

Definition

If C is a piecewise smooth curve, that is it is the union of smooth curves $C_1, C_2, \dots, C_n$ where the initial point of C_i is the end point of C_{i-1}, then we define

$$\int_C f(x, y)\ ds = \sum_{i=1}^{n} \int_{C_i} f(x, y)\ ds.$$

Example

Integrate $f(x, y) = 2x$ along the piecewise smooth curve C given by first traveling from $(0, 0)$ to $(1, 1)$ along the parabola $y = x^2$ and then traveling along the line segment from $(1, 1)$ to $(1, 2)$.

Example

Suppose that a wire lies along the upper half of the unit circle and has linear density at any point (x, y) proportional to its distance from the line $y = 1$. Compute the mass and center of mass of the wire.

Definition

Two other line integrals which occur naturally in many settings are the line integrals of $f(x, y)$ with respect to x and the line integral of $f(x, y)$ with respect to y. These are defined as

$$\begin{aligned} \int_C f(x, y)\ \mathrm{dx} &= \int_a^b f(x(t), y(t))x'(t)\ \mathrm{dt} \quad \text{and} \\ \int_C f(x, y)\ \mathrm{dy} &= \int_a^b f(x(t), y(t))y'(t)\ \mathrm{dt}. \end{aligned}$$

Notation

Since the line integrals of functions with respect to x and y frequently appear together, we often write

$$\int_C P(x, y)\ \mathrm{dx} + \int_C Q(x, y)\ \mathrm{dy} = \int_C P(x, y)\ \mathrm{dx} + Q(x, y)\ \mathrm{dy}.$$

Example

Evaluate $\int_C y^2 \, dx + x \, dy$ first where C is the line segment from $(-5, -3)$ to $(0, 2)$ and again where C is the arc from $(-5, -3)$ to $(0, 2)$ along the parabola $x = 4 - y^2$. Are the answers the same?

Note

If C denotes a curve segment parametrized by $r(t), a \leq t \leq b$, then we denote by $-C$ the same curve with opposite orientation, that is $-C$ is parametrized by $s(t) = r((b - t) + a)$.

Fact

$$\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx,$$
$$\int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy$$
$$\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds.$$

This is because Δx and Δy can be negative when orientation is reversed as in single variable calculus. However, Δs was defined to be an arc length which is positive. Thus the line integral with respect to arc length is independent to orientation.

Definition

If $f(x,y,z)$ is a function of 3 variables and C is a smooth curve in $\mathbb{R}^3$ parametrized by $r(t) = [x(t), y(t), z(t)]$ for $a \leq t \leq b$, then we define the line integral of f with respect to arc length as

$$\begin{aligned} \int_C f(x,y,z)\,\mathrm{ds} &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1}, y_{i-1}, z_{i-1}) \Delta s_i \\ &= \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,\mathrm{dt} \\ &= \int_a^b f(r(t))|r'(t)|\,\mathrm{dt}. \end{aligned}$$

Example

Evaluate $\int_C y\sin(z)\,\mathrm{ds}$ where C is the curve parametrized by $r(t) = [\cos(t), \sin(t), t]$ for $0 \leq t \leq 2\pi$.

Note

Suppose that F is a continuous force field and that a particle is moved by F through a smooth curve C parametrized by $r(t)$, $a \leq t \leq b$. Then the work done by F is given by

$$W = \int_C F \cdot T \text{ ds},$$

where $T(t)$ is the unit tangent vector of C at $r(t)$.

Definition

Let F be a continuous vector field defined on a smooth curve C parametrized by $r(t)$ for $a \leq t \leq b$. Let $T(t)$ denote the unit tangent vector of $r(t)$. Then the line integral of F along C is

$$\int_C F \cdot \mathrm{dr} = \int_a^b F(r(t)) \cdot r'(t)\ \mathrm{dt} = \int_C F \cdot T\ \mathrm{ds}.$$

Example

Evaluate $\int_C F \cdot \mathrm{dr}$ where $F(x, y, z) = [xy, yz, zx]$ and C is parametrized by $r(t) = [t, t^2, t^3]$ for $0 \leq t \leq 1$.

Note

Suppose that $F(x,y,z) = Pi + Qj + Rk$ is a vector field and that C is parametrized by $r(t) = [x(t), y(t), z(t)]$. Then

$$\int_C F \cdot \, \mathrm{dr} = \int_C P \, \mathrm{dx} + Q \, \mathrm{dy} + R \, \mathrm{dz}.$$