MTHSC 206 Section 17.3 – The Fundamental Theorem of Line Integrals

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Theorem

Let C be a smooth curve parametrized by $r(t)$ for $a \leq t \leq b$. Let f be a differentiable function whose gradient ∇f is continuous on C. Then

$$\int_C \nabla f \cdot \, dr = f(r(b)) - f(r(a)).$$

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Proof.

Let $g(t) = f(r(t))$. Then g is a real valued function of one variable and $g'(t) = \nabla f \cdot r'(t)$. So the theorem follows from the fundamental theorem of calculus. □

Definition

Suppose that F is a continuous vector field with domain D. We say that $\int_C F \cdot \, \mathrm{dr}$ is <u>independent of path</u> if $\int_{C_1} F \cdot \, \mathrm{dr} = \int_{C_2} F \cdot \, \mathrm{dr}$ for any two two paths C_1 and C_2 in D with the same initial and ending points.

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Definition

A curve C whose initial point and ending point are the same is called a <u>closed curve</u>.

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$\int_C F \cdot \, dr$ *is independent of path in* D *if and only if* $\int_C F \cdot \, dr = 0$ *for all closed curves.*

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Suppose that C_1 is a curve with initial point A and ending point B and that C_2 is a curve with initial point B and ending point A. Then C_1 and $-C_2$ have the same initial and ending points and $C = C_1 \cup C_2$ is a closed curve.

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Then C_1 and $-C_2$ have the same initial and ending points and $C = C_1 \cup C_2$ is a closed curve.
Further, we have

$$\int_C F\cdot\ \mathrm{dr} = \int_{C_1} F\cdot\ \mathrm{dr} + \int_{C_2} F\cdot\ \mathrm{dr} = \int_{C_1} F\cdot\ \mathrm{dr} - \int_{-C_2} F\cdot\ \mathrm{dr}.$$

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Idea of proof.

Take $f(x, y) = \int_{(a,b)}^{(x,y)} F \cdot$ dr where (a, b) is any point of D. Note that since D is connected, there is a path C from (a, b) to (x, y) for any point $(x, y) \in D$. Since, $\int_C F \cdot$ dr is path independent, it does not matter which path we choose. □

Theorem

If $F(x, y) = [P(x, y), Q(x, y)]$ is a conservative vector field (-i.e. $F = \nabla f$ for some $f(x, y)$.) where P and Q have continuous first order partial derivatives on D then throughout D we have

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Proof.

This follows from Clairaut's theorem. □

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Theorem

Let $F = [P, Q]$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first order partial derivatives and $P_y = Q_x$ throughout D. Then F is conservative.

Example

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Find the potential function for the field in the previous example.

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$P(r(a)) - P(r(b))$, which implies that
$K(r(b)) + P(r(b)) = K(r(a)) + P(r(a))$.