

# MTHSC 206 SECTION 17.4 – GREEN'S THEOREM

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## THEOREM

Let  $C$  be a positively oriented, piecewise smooth curve, simple closed curve in  $\mathbb{R}^2$ . Let  $D$  be the region bounded by  $C$ . If  $P(x, y)$  and  $Q(x, y)$  have continuous partial derivatives on an open region containing  $D$ , then

$$\int_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA.$$

## NOTE

In the situation above, we sometimes denote  $C$  as  $\partial D$ .

## SKETCH OF PROOF

We will assume that the region  $D$  is both of type I and type II.

Note that it is enough to show that

$$\int_C P \, dx = - \int \int_D \frac{\partial P}{\partial y} \, dA, \quad \text{and} \quad \int_C Q \, dy = \int \int_D \frac{\partial Q}{\partial x} \, dA.$$

We will show that first equality.

Writing

$$D = \{(x, y) \mid a \leq x \leq b; g_1(x) \leq y \leq g_2(x)\},$$

we have

$$\begin{aligned} \int \int_D \frac{\partial P}{\partial y} \, dA &= \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} \, dy \, dx \\ &= \int_a^b [P(x, g_2(x)) - P(x, g_1(x))] \, dx. \end{aligned}$$

## SKETCH OF PROOF CONTINUED ...

Now we break  $C$  into 4 curves  $C_1, C_2, C_3$  and  $C_4$  given by:

$$\begin{aligned}C_1 &: [t, g_1(t)]; \quad a \leq t \leq b \\C_2 &: [b, t]; \quad g_1(b) \leq t \leq g_2(b) \\-C_3 &: [t, g_2(t)]; \quad a \leq t \leq b \\-C_4 &: [a, t]; \quad g_1(a) \leq t \leq g_2(a).\end{aligned}$$

Note that  $\int_{C_2} P \, dx = 0 = \int_{C_4} P \, dx$ . Thus,

$$\begin{aligned}\int_C P \, dx &= \int_{C_1} P \, dx - \int_{-C_3} P \, dx \\&= \int_a^b [P(t, g_1(t)) - P(t, g_2(t))] \, dt \\&= - \int \int_D \frac{\partial P}{\partial y} \, dA.\end{aligned}$$

### EXAMPLE

Evaluate  $\int_C x^3 dx + xy dy$  where  $C$  is the curve bounding the triangular region with vertices  $(0, 0)$ ,  $(1, 0)$  and  $(0, 2)$ .

### EXAMPLE

Evaluate  $\int_C (3y - e^{\sin(x)}) dx + (7x + \sqrt{y^4 + 1}) dy$  where  $C$  is the circle about the origin of radius 3.

### NOTE

If  $P$  and  $Q$  are known to be zero on  $C$  and if  $D$  is the interior of  $C$  then no matter the behavior of  $P$  and  $Q$  in  $D$ , we have

$$\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \int_C P dx + Q dy = 0.$$

## NOTE

We can use Green's theorem to calculate area. We simply need to arrange for  $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ .

Here are some possible choices:

$$\begin{array}{lll} P(x, y) = 0 & P(x, y) = -y, & P(x, y) = \frac{-y}{2} \\ Q(x, y) = x & Q(x, y) = 0, & Q(x, y) = \frac{x}{2}. \end{array}$$

Then, Green's theorem gives

$$A = \int_C x \, dy = - \int_C y \, dx = \frac{1}{2} \int_C x \, dy - y \, dx.$$

### EXAMPLE

Find the area enclosed by the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2$ .

### NOTE

We can use Green's theorem to integrate over regions which are not of type I and of type II but are finite unions of such regions.

### EXAMPLE

Evaluate  $\int_C y^2 dx + 3xy dy$  where  $C$  is the boundary of the region bounded above by the upper semicircle of radius 2 and below by the upper semicircle of radius 1.

## NOTE

With some care, Green's theorem can be extended to regions with holes.

## EXAMPLE

If  $F(x, y) = \left[ \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right]$ , show that  $\int_C F \cdot dr = 2\pi$  for every positively oriented, simple, closed path that encloses the origin.