MTHSC 206 Section 17.4 – Green's Theorem

Kevin James

Theorem

Let C be a positively oriented, piecewise smooth curve, simple closed curve in $\mathbb{R}^2$. Let D be the region bounded by C. If $P(x, y)($ and $Q(x, y)$ have continuous partial derivatives on an open region containing D, then

$$\int_C P\ dx + Q\ dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA.$$

Theorem

Let C be a positively oriented, piecewise smooth curve, simple closed curve in $\mathbb{R}^2$. Let D be the region bounded by C. If $P(x, y)($ and $Q(x, y)$ have continuous partial derivatives on an open region containing D, then

$$\int_C P\ dx + Q\ dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA.$$

Note

In the situation above, we sometimes denote C as ∂D.

Sketch of Proof

We will assume that the region D is both of type I and type II.

Sketch of Proof

We will assume that the region D is both of type I and type II. Note that it is enough to show that
$\int_C P \,\mathrm{dx} = -\int\int_D \frac{\partial P}{\partial y} \,\mathrm{dA}, \quad \text{and} \quad \int_C Q \,\mathrm{dy} = \int\int_D \frac{\partial Q}{\partial x} \,\mathrm{dA}.$

Sketch of Proof

We will assume that the region D is both of type I and type II.
Note that it is enough to show that
$\int_C P \text{ dx} = -\int\int_D \frac{\partial P}{\partial y} \text{ dA}, \quad \text{and} \quad \int_C Q \text{ dy} = \int\int_D \frac{\partial Q}{\partial x} \text{ dA}.$
We will show that first equality.

Sketch of Proof

We will assume that the region D is both of type I and type II.
Note that it is enough to show that
$\int_C P \text{ dx} = -\int\int_D \frac{\partial P}{\partial y} \text{ dA}, \quad \text{and} \quad \int_C Q \text{ dy} = \int\int_D \frac{\partial Q}{\partial x} \text{ dA}.$
We will show that first equality.
Writing

$$D = \{(x, y) \mid a \leq x \leq b; g_1(x) \leq y \leq g_2(x)\},$$

we have

$$\begin{aligned} \int\int_D \frac{\partial P}{\partial y} \text{ dA} &= \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} \text{ dy dx} \\ &= \end{aligned}$$

Sketch of Proof

We will assume that the region D is both of type I and type II.
Note that it is enough to show that
$\int_C P \, dx = -\int\int_D \frac{\partial P}{\partial y} \, dA,$ and $\int_C Q \, dy = \int\int_D \frac{\partial Q}{\partial x} \, dA.$
We will show that first equality.
Writing

$$D = \{(x, y) \mid a \le x \le b; g_1(x) \le y \le g_2(x)\},$$

we have

$$\begin{aligned} \int\int_D \frac{\partial P}{\partial y} \, dA &= \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} \, dy \, dx \\ &= \int_a^b [P(x, g_2(x)) - P(x, g_1(x))] \, dx. \end{aligned}$$

Sketch of proof continued ...

Now we break C into 4 curves $C1$, $C2$, $C3$ and $C4$ given by:

$$\begin{aligned} C1 &: [t, g_1(t)]; \quad a \le t \le b \\ C2 &: [b, t]; \quad g_1(b) \le t \le g_2(b) \\ -C3 &: [t, g_2(t)]; \quad a \le t \le b \\ -C4 &: [a, t]; \quad g_1(a) \le t \le g_2(a). \end{aligned}$$

Sketch of proof continued ...

Now we break C into 4 curves $C1$, $C2$, $C3$ and $C4$ given by:

$$\begin{aligned} C1 &: [t, g_1(t)]; \quad a \le t \le b \\ C2 &: [b, t]; \quad g_1(b) \le t \le g_2(b) \\ -C3 &: [t, g_2(t)]; \quad a \le t \le b \\ -C4 &: [a, t]; \quad g_1(a) \le t \le g_2(a). \end{aligned}$$

Note that $\int_{C2} P \; \mathrm{dx} = 0 = \int_{C4} P \; \mathrm{dx}$. Thus,

Sketch of proof continued ...

Now we break C into 4 curves $C1$, $C2$, $C3$ and $C4$ given by:

$$\begin{aligned} C1 &: [t, g_1(t)]; \quad a \leq t \leq b \\ C2 &: [b, t]; \quad g_1(b) \leq t \leq g_2(b) \\ -C3 &: [t, g_2(t)]; \quad a \leq t \leq b \\ -C4 &: [a, t]; \quad g_1(a) \leq t \leq g_2(a). \end{aligned}$$

Note that $\int_{C2} P \text{ dx} = 0 = \int_{C4} P \text{ dx}$. Thus,

$$\int_C P \text{ dx} = \int_{C1} P \text{ dx} - \int_{-C3} P \text{ dx}$$

Sketch of proof continued ...

Now we break C into 4 curves $C1$, $C2$, $C3$ and $C4$ given by:

$$\begin{aligned} C1 &: [t, g_1(t)]; \quad a \le t \le b \\ C2 &: [b, t]; \quad g_1(b) \le t \le g_2(b) \\ -C3 &: [t, g_2(t)]; \quad a \le t \le b \\ -C4 &: [a, t]; \quad g_1(a) \le t \le g_2(a). \end{aligned}$$

Note that $\int_{C2} P\ \mathrm{dx} = 0 = \int_{C4} P\ \mathrm{dx}$. Thus,

$$\begin{aligned} \int_C P\ \mathrm{dx} &= \int_{C1} P\ \mathrm{dx} - \int_{-C3} P\ \mathrm{dx} \\ &= \end{aligned}$$

Sketch of proof continued ...

Now we break C into 4 curves $C1$, $C2$, $C3$ and $C4$ given by:

$$\begin{aligned} C1 &: [t, g_1(t)]; \quad a \le t \le b \\ C2 &: [b, t]; \quad g_1(b) \le t \le g_2(b) \\ -C3 &: [t, g_2(t)]; \quad a \le t \le b \\ -C4 &: [a, t]; \quad g_1(a) \le t \le g_2(a). \end{aligned}$$

Note that $\int_{C2} P \text{ dx} = 0 = \int_{C4} P \text{ dx}$. Thus,

$$\begin{aligned} \int_C P \text{ dx} &= \int_{C1} P \text{ dx} - \int_{-C3} P \text{ dx} \\ &= \int_a^b [P(t, g_1(t)) - P(t, g_2(t))] \text{ dt} \end{aligned}$$

Sketch of proof continued ...

Now we break C into 4 curves $C1$, $C2$, $C3$ and $C4$ given by:

$$
\begin{aligned}
C1 &: \quad [t, g_1(t)]; \quad a \le t \le b \\
C2 &: \quad [b, t]; \quad g_1(b) \le t \le g_2(b) \\
-C3 &: \quad [t, g_2(t)]; \quad a \le t \le b \\
-C4 &: \quad [a, t]; \quad g_1(a) \le t \le g_2(a).
\end{aligned}
$$

Note that $\int_{C2} P \text{ dx} = 0 = \int_{C4} P \text{ dx}$. Thus,

$$
\begin{aligned}
\int_C P \text{ dx} &= \int_{C1} P \text{ dx} - \int_{-C3} P \text{ dx} \\
&= \int_a^b [P(t, g_1(t)) - P(t, g_2(t))] \text{ dt} \\
&= -\int\int_D \frac{\partial P}{\partial y} \text{ dA}.
\end{aligned}
$$

Sketch of proof continued ...

Now we break C into 4 curves $C1$, $C2$, $C3$ and $C4$ given by:

$$\begin{aligned}
C1 &: [t, g_1(t)]; \quad a \leq t \leq b \\
C2 &: [b, t]; \quad g_1(b) \leq t \leq g_2(b) \\
-C3 &: [t, g_2(t)]; \quad a \leq t \leq b \\
-C4 &: [a, t]; \quad g_1(a) \leq t \leq g_2(a).
\end{aligned}$$

Note that $\int_{C2} P \, \mathrm{dx} = 0 = \int_{C4} P \, \mathrm{dx}$. Thus,

$$\begin{aligned}
\int_C P \, \mathrm{dx} &= \int_{C1} P \, \mathrm{dx} - \int_{-C3} P \, \mathrm{dx} \\
&= \int_a^b \left[P(t, g_1(t)) - P(t, g_2(t))\right] \, \mathrm{dt} \\
&= -\int\int_D \frac{\partial P}{\partial y} \, \mathrm{dA}.
\end{aligned}$$

Example

Evaluate $\int_C x^3 \, dx + xy \, dy$ where C is the curve bounding the triangular region with vertices $(0, 0)$, $(1, 0)$ and $(0, 2)$.

Example

Evaluate $\int_C x^3 \, \mathrm{dx} + xy \, \mathrm{dy}$ where C is the curve bounding the triangular region with vertices $(0,0)$, $(1,0)$ and $(0,2)$.

Example

Evaluate $\int_C (3y - e^{\sin(x)}) \, \mathrm{dx} + (7x + \sqrt{y^4 + 1}) \, \mathrm{dy}$ where C is the circle about the origin of radius 3.

Example

Evaluate $\int_C x^3 \, \mathrm{dx} + xy \, \mathrm{dy}$ where C is the curve bounding the triangular region with vertices $(0,0)$, $(1,0)$ and $(0,2)$.

Example

Evaluate $\int_C (3y - e^{\sin(x)}) \, \mathrm{dx} + (7x + \sqrt{y^4 + 1}) \, \mathrm{dy}$ where C is the circle about the origin of radius 3.

Note

If P and Q are known to be zero on C and if D is the interior of C then no matter the behavior of P and Q in D, we have
$\int \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \int_C P \, \mathrm{dx} + Q \, \mathrm{dy} = 0.$

Note

We can use Green's theorem to calculate area. We simply need to arrange for $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$.

Note

We can use Green's theorem to calculate area. We simply need to arrange for $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$.
Here are some possible choices:

$$\begin{aligned} P(x,y) &= 0 \qquad & P(x,y) &= -y, \quad & P(x,y) &= \frac{-y}{2} \\ Q(x,y) &= x \qquad & Q(x,y) &= 0, \quad & Q(x,y) &= \frac{x}{2}. \end{aligned}$$

Note

We can use Green's theorem to calculate area. We simply need to arrange for $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$.
Here are some possible choices:

$$\begin{aligned} P(x,y) &= 0 \qquad & P(x,y) &= -y, \quad & P(x,y) &= \frac{-y}{2} \\ Q(x,y) &= x \qquad & Q(x,y) &= 0, \quad & Q(x,y) &= \frac{x}{2}. \end{aligned}$$

Then, Green's theorem gives

$$A = \int_C x \; \mathrm{dy} = -\int_C y \; \mathrm{dx} = \frac{1}{2}\int_C x \; \mathrm{dy} - y \; \mathrm{dx}.$$

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2$.

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2$.

Note

We can use Green's theorem to integrate over regions which are not of type I and of type II but are finite unions of such regions.

Example

Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2$.

Note

We can use Green's theorem to integrate over regions which are not of type I and of type II but are finite unions of such regions.

Example

Evaluate $\int_C y^2 \, \mathrm{d}x + 3xy \, \mathrm{d}y$ where C is the boundary of the region bounded above by the upper semicircle of radius 2 and below by the upper semicircle of radius 1.

Note

With some care, Green's theorem can be extended to regions with holes.

Note

With some care, Green's theorem can be extended to regions with holes.

Example

If $F(x,y) = \left[\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right]$, show that $\int_C F \cdot \, \mathrm{dr} = 2\pi$ for every positively oriented, simple, closed path that encloses the origin.