MTHSC 206 Section 17.5 – Curl and Divergence

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Definition

Suppose that $F = [P, Q, R]$ is a vector field on $\mathbb{R}^3$. We define the curl of F as

$$\text{curl}(F) = \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right), \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right), \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right]$$

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Example

Let $F = [xz, xyz, z^2]$. Find $\operatorname{curl}(F)$.

Theorem

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Example

Is the vector field $F(x, y, z) = [xz, xyz, -y^2]$ conservative?

Theorem

Suppose that F is a vector field on $\mathbb{R}^3$ all of whose partial derivatives are continuous and that $curl(F) = 0$. Then, F is conservative. (Actually, we only need the domain of F to be simply connected for the result to hold).

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Example

Let $F(x, y, z) = [y^2z^3, 2xyz^3, 3xy^2z^2]$. Show that F is conservative and find its potential function.

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Suppose that $F(x, y, z) = [P, Q, R]$ *is a vector field on* $\mathbb{R}^3$ *and that* P, Q, *and* R *all have continuous second order partials. Then,*

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Suppose that $F(x, y, z) = [P, Q, R]$ *is a vector field on* $\mathbb{R}^3$ *and* $div(F) \neq 0$. *Then* F *is not the curl of another vector field.*

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Corollary

Suppose that $F(x,y,z) = [P,Q,R]$ is a vector field on $\mathbb{R}^3$ and $div(F) \neq 0$. Then F is not the curl of another vector field.

Example

Show that $F(x,y,z) = [xz, xyz, -y^2]$ is not the curl of another vector field.

Theorem

Suppose that $F(x, y) = [P, Q]$ is a vector field on $\mathbb{R}^2$. We can think of F as a vector field on $\mathbb{R}^3$ defined by $F(x, y, z) = [P(x, y), Q(x, y), 0]$. With this interpretation we have

$$\int_C F \cdot \; dr = \int \int_D curl(F) \cdot K \; dA$$

$$\int_C F \cdot n \; ds = \int \int_D div(F)(x, y) \; dA$$

where C is the boundary curve of the region D, $r(t)$ parametrizes C and n is the normal vector to r, and we assume that D and C satisfy the hypotheses of Green's theorem.