MTHSC 206 Section 17.6 – Parametric Surfaces and their Areas

Kevin James

Note

As we parametrized space curves with a vector valued function $r(t)$ of one variable, we can parametrize a surface (-i.e. a tow dimensional object in $\mathbb{R}^3$) with a vector valued function $r(u, v)$ of two variables.

If there is a vector valued function $r(u, v) = [x(u, v), y(u, v), z(u, v)]$ such that the surface S is traced out by r as (u, v) varies over some region D, then we call S a parametric surface.

We refer to the equations $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$ as the parametric equations of the surface.

Example

Identify the surface parametrized by
$r(u, v) = [2\cos(u), v, 2\sin(u)]$.

Note

In order to get an idea of the graph of a parametric surface, it is often useful to hold either u or v constant and let the other vary. The functions $r(u, v_0)$ and $r(u_0, v)$ where u_0 and v_0 are constants are vector valued functions of one variable and thus trace out space curves. These curves are referred to as grid curves.

Example

What are the grid curves in the above example?

Example

Find a parametrization of the plane passing through the point P_0 with position vector r_0 and containing two nonparallel vectors $\vec{a}$ and $\vec{b}$.

Example

Find a parametric representation of the sphere $x^2 + y^2 + z^2 = a^2$.

Note

Parametrizations of a surface are not unique.

Example

Find two parametrizations of the cone $z = 2\sqrt{x^2 + y^2}$.

Fact

If S is the surface formed by rotating the graph of $y = f(x)$ above $[a, b]$ about the x-axis, then we can parametrize S by

$$r(x, \theta) = [x, f(x)\cos(\theta), f(x)\sin(\theta)], \quad a \leq x \leq b; \quad 0 \leq \theta \leq 2\pi.$$

Example

Find the parametric equations for the surface generated by rotating the curve $y = x^2$, $0 \leq x \leq 2$ about the x-axis.

Definition

Suppose that S is a surface parametrized by $r(u,v)$, $(u,v) \in D$. We say that S is smooth if $r_u = [\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}]$ and $r_v = [\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}]$ satisfy $r_u \times r_v \neq 0$ for all $(u,v) \in D$.

Fact

If S is a smooth surface parametrized by $r(u,v)$, then the tangent plane to S at the point $r(u_0,v_0)$ is the plane containing the point $r(u_0,v_0)$ and the vectors $r_u(u_0,v_0)$ and $r_v(u_0,v_0)$.
This plane has normal vector $n = r_u(u_0,v_0) \times r_v(u_0,v_0)$.

Example

Find the tangent plane to the surface parametrized by $r(u,v) = [u^2, v^2, u + 2v]$ at $(1,1,3)$.

Definition

Suppose that a smooth surface S is parametrized by $r(u, v) = [x(u, v), y(u, v), z(u, v)]$, $(u, v) \in D$ and that S is covered just once as (u, v) traverses D. Then the surface area of S is

$$A(S) = \int\int_D |r_u \times r_v| \, dA.$$

Example

Find the surface area of the sphere of radius a.

Definition

Suppose that S is the graph of a function $f(x, y)$ then S is parametrized by $r(u, v) = [u, v, f(u, v)]$ and the surface area of S is

$$A(S) = \int\int_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA.$$

Example

Find the area of the part of the paraboloid $z = x^2 + y^2$ which lies below the plane $z = 9$.