MTHSC 206 Section 17.8 – Stokes' Theorem

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Goal

We wish to generalize Green's theorem to surfaces. That is we wish to relate the integral of a function over a surface to the line integral of some function around the boundary.

Induced Orientation

Given a smooth orientable surface S and an orientation (-i.e. a continuous choice of unit normal on S), we define the orientation on the boundary ∂S of S to be such that if you walk around C in the positive direction with your head pointing in the direction of the unit normal of S, then then the surface will remain on your left.

Theorem (Stokes)

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve ∂S with orientation induced form S. Let F be a vector field whose components have continuous partials on an open region of $\mathbb{R}^3$ that contains S. Then

$$\int\int_C F \cdot\ dr = \int\int_S curl(F) \cdot\ d\mathbb{S}$$

Example

Evaluate $\int_C F \cdot \, \mathrm{dr}$ where $F(x, y, z) = [-y^2, x, z^2]$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.

Example

Use Stokes' Theorem to compute the integral $\int \int_S \, \mathrm{curl}(F) \cdot \, d\mathbb{S}$, where $F = [xz, yz, xy]$ and S is the part of the sphere of radius 2 lying inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.

Note

Note that if S_1 and S_2 are two orientable piecewise-smooth surfaces with the same boundary C which satisfies the hypothesis of Stokes' theorem, then

$$\int\int_{S_1} \text{curl}(F)\ d\mathbb{S} = \int\int_{C} F\cdot\ \text{dr} = \int\int_{S_2} \text{curl}(F)\cdot\ d\mathbb{S}.$$

Note

If F is conservative then $\text{curl}(F) = 0$. Thus for piecewise–smooth closed curves C we have

$$\int\int_{C} F\cdot\ \text{dr} = \int\int_{S} \text{curl}(F)\cdot\ d\mathbb{S} = 0,$$

where S is any smooth orientable surface with $\partial S = C$.