MTHSC 3110 Section 1.1

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A system of linear equations is a collection of equations in the same set of variables.
For example,

$$\begin{cases} x_1 + 3x_2 & = 5 \\ 2x_1 - x_2 & = -4 \end{cases}$$

Of course, since this is a pair of equations in two variables we could think of this as the intersection of two lines.

Exercise

Sketch these lines and find their intersection.

Solution Sets

The solution set for a linear system of equations can be described as

- Empty. (-i.e. There is no solution.)
- Exactly one solution
- More than one solution

It is easy to come up with examples of the first two circumstances. The third possibility actually means that something much stronger is true: if we have two distinct solutions then we must have infinitely many solutions.

Exercise

Draw two dimensional examples illustrating each of the three possible outcomes.
Why is it that if we have at least two solutions then there are infinitely many?

Matrix Notation

The system of equations above has coefficient matrix

$$\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$$

and augmented matrix

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & -1 & -4 \end{pmatrix}$$

Definition

A matrix with m rows and n columns is referred to as an $m \times n$ matrix.

Note

the number of rows always comes before the number of columns. Thus the matrices above are 2×2 and 2×3 respectively.

Elementary row operations

Exercise

Subtract twice the first row from the second row of the augmented matrix. Explain why the new system of equations has precisely the same set of solutions as the old set of equations.

Definition

There are 3 types of elementary row operations.

1. (Replacement) Replace a row by the sum of *the same row* and a multiple of *different* row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply a row by a non-zero constant.

Notation

1. (Replacement) The operation of replacing row i by itself plus c times row j will be denoted $R_i + cR_j$.
2. (Interchange) The operation of interchanging rows i and j will be denoted by $R_i \leftrightarrow R_j$.
3. (Scaling) The operation of scaling row i by a nonzero constant c will be denoted by cR_i.

Note

It is clear that the set of solutions is not changed by the second and third operations.

Exercise

Explain why the set of solutions is unchanged by the first operation.

Equivalence of Linear Systems

Definition

1. Two linear systems are equivalent if they have the same solution set.
2. Two matrices are row equivalent if one matrix can be obtained from the other by elementary row operations.

Exercise

Suppose A and B are $m \times n$ matrices: explain why the following is true: if A can be reduced via elementary row operations to B, then B can be reduced to A.

Note

The key fact to notice is that these operations can be undone or inverted.

Fact

Two linear systems are equivalent if and only if their augmented matrices are row-equivalent.

Exercise

Write the following system of equations as an augmented matrix and solve the system:

$$\begin{array}{ccccccc} x_1 & + & & + & -3x_3 & = & 8 \\ 2x_1 & + & 2x_2 & + & 9x_3 & = & 7 \\ & + & x_2 & + & 5x_3 & = & -2 \end{array}$$

Fundamental Questions

We can rephrase our question about how many solutions there are to a system in the following way.

Question

1. Is the system consistent, that is, do there exist *any* solutions?
2. If there is at least one solution, is it unique?

Example

For example, we've seen that the system

$$\begin{aligned} x_1 + 3x_2 &= 5 \\ 2x_1 - x_2 &= -4 \end{aligned}$$

has a unique solution. On the other hand, the system

$$\begin{aligned} x_1 + 3x_2 &= 5 \\ 2x_1 + 6x_2 &= -4 \end{aligned}$$

has no solutions, and the system

$$\begin{aligned} x_1 + 3x_2 &= 5 \\ 2x_1 + 6x_2 &= 10 \end{aligned}$$

has infinitely many.