Row Reduction and Row Echelon Form

Definition

A rectangular matrix is in echelon form (or row echelon form) if it has the following properties

1. All non-zero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

Sometimes we want more than echelon form. We can make all the leading entries 1 by multiplying by a constant, and we can subtract from rows above to zero out their entries in that column.

Definition

A rectangular matrix is in reduced echelon form (or reduced row echelon form) if it is in row echelon form, and has the following additional properties

1. The leading non-zero term of every non-zero row is 1
2. Each leading 1 is the only non-zero entry in its column.

Examples

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is in row echelon form but is not reduced:

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is in reduced row echelon form:

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is not in row echelon form

Solving a system in reduced row echelon form

Suppose we have a system of equations, we've written them as an augmented matrix, we've performed elementary row operations, and arrived at the following reduced row echelon form matrix.

$$\begin{pmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

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Pair up the variables and the pivot columns: these are the <u>basic variables</u>. The remaining variables are the <u>free variables</u>.

Great News

Theorem

Each matrix is row equivalent to one and only one reduced echelon matrix.

Gaussian elimination

Question

How do we get a matrix into row echelon or reduced row echelon form?

Gaussian Elimination

We start with the left-most non-zero column, working to the right and from the top down. At each stage, we will be working with the portion of the matrix which is below or to the right (or both) of the pivot.

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3. Use row replacement operations to change all the values below the pivot to zero.

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Once we've gone through all the rows, the matrix is in row echelon form.

Gaussian Elimination continued

5. Scale pivots to be 1

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6. Use row replacement operations to change all the values above pivots to be zero. (There are technical reasons for doing this from the bottom pivot first).

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Definition

This process is often referred to as Gaussian elimination, or Gauss-Jordan elimination

Example

Exercise

Row reduce

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{pmatrix} \to \cdots$$

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3. Infinitely many solutions: for example

$$\begin{pmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

General Form

In the last case, we write the solution set as

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Note

1. This is called a *general solution*.
2. x_1 and x_3 are called dependent variables.
3. x_2 is called a parameter or free variable or an independent variable.

Exercise

Find the general solution to the linear system with augmented matrix

$$\begin{pmatrix} 1 & 6 & 2 & -5 & -1 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix} \to \cdots$$

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Solution Procedures (p24)

Given a linear system to solve

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Solution Procedures (p24)

Given a linear system to solve

1. Write the augmented matrix
2. Perform row reduction to obtain echelon form. If the system is not consistent then there are no solutions and you may stop.

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4. Write system of equations corresponding to reduced echelon form.

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5. Basic variables correspond to columns with pivots. Free variables correspond to columns without pivots.

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4. Write system of equations corresponding to reduced echelon form.
5. Basic variables correspond to columns with pivots. Free variables correspond to columns without pivots.
6. Write basic variables in terms of free variables.