MTHSC 412 Section 1.3 – Vector Arithmetic

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Definition

A *vector* is a matrix with one column.

Example

$$\begin{pmatrix} 1 \\ 2 \\ -5 \\ 9 \end{pmatrix}$$

Note

Two vectors are equal precisely when they have the same number of rows and all their corresponding entries are equal.

Definition

We define the sum and of two vectors by

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix}$$

and the product of a scalar and a vector by

$$\alpha \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \\ \vdots \\ \alpha u_n \end{pmatrix}$$

Example

$$\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} \qquad \text{and} \qquad 3\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ 6 \\ 3 \end{pmatrix}$$

Exercise

Let $\vec{u}$ and $\vec{v}$ be given by

$$\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Plot $\vec{u}$, $\vec{v}$, $2\vec{u}$ and $\vec{u} + \vec{v}$.

Parallelogram rule for vector addition

Suppose $\vec{u}$ and $\vec{v} \in \mathbb{R}^2$. Then $\vec{u} + \vec{v}$ corresponds to the fourth vertex of the parallelgram whose opposite vertex is $\vec{0}$ and whose other two vertices are $\vec{u}$ and $\vec{v}$.

Exercise

Let $\vec{u} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$. Display $\vec{u}$, $-2/3\vec{u}$, $\vec{v}$ and $-2/3\vec{u} + \vec{v}$ on a graph.

In general we will consider vectors in $\mathbb{R}^n$, that is, having n real entries. $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n$

The zero vector is $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ having n entries, each equal to 0.

THEOREM

Suppose that $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ *and* $c, d \in \mathbb{R}$. *Then,*

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3. $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$
4. $\vec{u} + -\vec{u} = -\vec{u} + \vec{u} = \vec{0}$ $\qquad$ $(-\vec{u} = (-1)\vec{u})$
5. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
6. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
7. $c(d\vec{u}) = (cd)\vec{u}$
8. $1 \cdot \vec{u} = \vec{u}$

Definition

Let p be a positive integer. Given vectors $\vec{u}_1, \vec{u}_2, \dots \vec{u}_p$ in $\mathbb{R}^n$, and $c_1, c_2, \dots c_p$ in $\mathbb{R}$, the vector

$$\vec{u} = c_1\vec{u}_1 + c_2\vec{u}_2 + \dots c_p\vec{u}_p$$

is called a linear combination of the vectors $\vec{u}_1, \dots \vec{u}_p$ with weights $c_1, \dots c_p$.

Example

$$2\begin{pmatrix}1\\-2\\7\end{pmatrix} + 3\begin{pmatrix}1\\1\\1\end{pmatrix} + -2\begin{pmatrix}2\\3\\-8\end{pmatrix} = \begin{pmatrix}\quad\\\quad\\\quad\end{pmatrix}$$

is a linear combination of $\begin{pmatrix}1\\-2\\7\end{pmatrix}$, $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\3\\-8\end{pmatrix}$ with weights $2, 3, -2$.

Geometry

Let $\vec{u} = \begin{pmatrix}1\\1\end{pmatrix}$ and $\vec{v} = \begin{pmatrix}1\\-1\end{pmatrix}$. Show all linear combinations of $\vec{u}$ and $\vec{v}$ on a graph.

Exercise

let $\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ $\vec{u}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$ Is $\vec{b}$ a linear combination of $\vec{u}_1$ and $\vec{u}_2$

Fact

A vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{b}$$

has the same solution set as the system of equations whose augmented matrix is

$$\left(\begin{array}{ccccc} | & | & \dots & | & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n & \vec{b} \\ | & | & \dots & | & | \end{array}\right)$$

In particular, $\vec{b}$ is a linear combination of $\vec{v}_1, \dots \vec{v}_n$ if and only if the system of linear equations is consistent.

Definition

Suppose $\vec{v}_1, \vec{v}_2, \dots \vec{v}_p \in \mathbb{R}^n$. We define

$$\mathrm{Span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots c_p\vec{v}_p : c_1, c_2, \dots, c_p \in \mathbb{R}\}$$

That is, $\mathrm{Span}(\vec{v}_1, \dots, \vec{v}_p)$ is the set of all linear combinations of $\vec{v}_1, \dots, \vec{v}_p$.

Note

1. The span of $\vec{0}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is the single point $\vec{0}$.
2. The span of a single non-zero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a line through $\vec{0}$.
3. The span of two non-zero vectors in $\mathbb{R}^3$ is either a plane through $\vec{0}$ or, if one vector is a scalar multiple of the other, a line through $\vec{0}$.

Exercise

Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -9 \\ -30 \\ 31 \end{pmatrix}$.

Span$(\vec{v}_1, \vec{v}_2)$ is a plane in $\mathbb{R}^3$ Is $\vec{b}$ in that plane?

Exercise

Read the application on p.31.