MTHSC 3110 Section 1.3 – Vector Arithmetic

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Note

Two vectors are equal precisely when they have the same number of rows and all their corresponding entries are equal.

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Exercise

Let $\vec{u}$ and $\vec{v}$ be given by

$$\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Plot $\vec{u}$, $\vec{v}$, $2\vec{u}$ and $\vec{u} + \vec{v}$.

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Parallelogram rule for vector addition

Suppose $\vec{u}$ and $\vec{v} \in \mathbb{R}^2$. Then $\vec{u} + \vec{v}$ corresponds to the fourth vertex of the parallelgram whose opposite vertex is $\vec{0}$ and whose other two vertices are $\vec{u}$ and $\vec{v}$.

Exercise

Let $\vec{u} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$. Display $\vec{u}$, $-2/3\vec{u}$, $\vec{v}$ and $-2/3\vec{u} + \vec{v}$ on a graph.

In general we will consider vectors in $\mathbb{R}^n$, that is, having n real entries. $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n$

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The zero vector is $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ having n entries, each equal to 0.

Theorem

Suppose that $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ *and* $c, d \in \mathbb{R}$. *Then,*

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3. $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$
4. $\vec{u} + -\vec{u} = -\vec{u} + \vec{u} = \vec{0}$ $\quad (-\vec{u} = (-1)\vec{u} \ )$
5. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
6. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
7. $c(d\vec{u}) = (cd)\vec{u}$
8. $1 \cdot \vec{u} = \vec{u}$

Definition

Let p be a positive integer. Given vectors $\vec{u}_1, \vec{u}_2, \dots \vec{u}_p$ in $\mathbb{R}^n$, and $c_1, c_2, \dots c_p$ in $\mathbb{R}$, the vector

$$\vec{u} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + \dots c_p \vec{u}_p$$

is called a linear combination of the vectors $\vec{u}_1, \dots \vec{u}_p$ with weights $c_1, \dots c_p$.

Example

$$2\begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + -2\begin{pmatrix} 2 \\ 3 \\ -8 \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix}$$

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Geometry

Let $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Show all linear combinations of $\vec{u}$ and $\vec{v}$ on a graph.

Exercise

let $\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ $\vec{u}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$ Is $\vec{b}$ a linear combination of $\vec{u}_1$ and $\vec{u}_2$

Fact

A vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{b}$$

has the same solution set as the system of equations whose augmented matrix is

$$\begin{pmatrix} | & | & \dots & | & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n & \vec{b} \\ | & | & \dots & | & | \end{pmatrix}$$

In particular, $\vec{b}$ is a linear combination of $\vec{v}_1, \dots \vec{v}_n$ if and only if the system of linear equations is consistent.

Definition

Suppose $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p \in \mathbb{R}^n$. We define

$$\text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots c_p\vec{v}_p : c_1, c_2, \ldots, c_p \in \mathbb{R}\}$$

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That is, $\text{Span}(\vec{v}_1, \ldots, \vec{v}_p)$ is the set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_p$.

Note

1. The span of $\vec{0}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is the single point $\vec{0}$.
2. The span of a single non-zero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a line through $\vec{0}$.
3. The span of two non-zero vectors in $\mathbb{R}^3$ is either a plane through $\vec{0}$ or, if one vector is a scalar multiple of the other, a line through $\vec{0}$.

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EXERCISE

Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -9 \\ -30 \\ 31 \end{pmatrix}$.

Span$(\vec{v}_1, \vec{v}_2)$ is a plane in $\mathbb{R}^3$ Is $\vec{b}$ in that plane?

Exercise

Read the application on p.31.