MTHSC 3110 Section 1.5 – Solution sets of linear equations

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Note

$\vec{x} = \vec{0}$ is always solution to $A\vec{x} = \vec{0}$. It is called the *trivial solution*.

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The homogeneous equation $A\vec{x} = \vec{0}$ has a non-trivial solution if and only if it has free variables.

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Thus we have nontrivial solutions. □

Exercise

1. Determine if the following homogeneous system has non-trivial solutions:

$$\begin{cases} 2x_1 + 3x_2 + x_3 = 0 \\ \phantom{2x_1 + {}} 5x_2 - x_3 = 0 \\ -x_1 + x_2 - x_3 = 0 \end{cases}$$

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2. Describe the solution set.

Note

The solution set of $A\vec{x} = \vec{0}$ can always be written as $\text{Span}(\vec{v}_1, \ldots, \vec{v}_p)$ for some vectors $\vec{v}_1, \ldots, \vec{v}_p$.

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Definition

An equation of the form

$$\vec{x} = s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_k\vec{v}_k$$

is said to be in *vector parametric form*.

Example

Describe the solution sets of $A\vec{x} = \vec{0}$ and $A\vec{x} = \vec{b}$ where

$$A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}$$

Theorem

Suppose that the equation $A\vec{x} = \vec{b}$ has a solution $\vec{p}$. Then all solutions to the equation have the form

$$\vec{w} = \vec{p} + \vec{v}_h$$

where $\vec{v}_h$ is a solution to the corresponding homogeneous equation $A\vec{x} = \vec{0}$.

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where $\vec{v}_h$ is a solution to the corresponding homogeneous equation $A\vec{x} = \vec{0}$.

Note

That is, if $\vec{p}$ is any solution to $A\vec{x} = \vec{b}$, and the solution set of $A\vec{x} = \vec{0}$ is $\text{Span}(\vec{v}_1, \ldots, \vec{v}_k)$, then the solution set of $A\vec{x} = \vec{b}$ is

$$\vec{p} + \text{Span}(\vec{v}_1, \ldots, \vec{v}_k)$$