MTHSC 3110 Section 1.7 – Linear Independence

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Definition

An indexed set of vectors $\{\vec{v}_1, \ldots, \vec{v}_p\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$x_1\vec{v}_1 + \cdots + x_p\vec{v}_p = \vec{0}$$

has only the trivial solution ($\vec{x} = \vec{0}$).
Otherwise if there exist $c_1, \ldots, c_p \in \mathbb{R}$ not all zero, so that

$$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$$

then the set is *linearly dependent*.

Example

Are the vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} -1 \\ 7 \\ 21 \end{pmatrix}$, linearly independent? If not, find a dependence.

Note

The columns of the matrix A are linearly independent if and only if the equation $A\vec{x} = \vec{0}$ has only the trivial solution.

Example

Are the columns of $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 2 \\ 5 & 3 & 3 \end{pmatrix}$ linearly idependent?

Note

1. A set containing a single vector $\vec{v}$ is linearly independent if and only if $\vec{v} \neq 0$.
2. A set containing two vectors is linearly independent if and only if neither vector is a multiple of the other.

Proof.

$\square$

Theorem

An indexed set $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ is linearly dependent if and only if one of the vectors in S is a linear combination of the others. In fact, S is linearly dependent if and only if either $\vec{v}_1 = \vec{0}$, or there is a j so that $\vec{v}_j$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}$.

Proof.

$\square$

Example

Let $\vec{u} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Describe $\mathrm{Span}(\vec{u}, \vec{v})$. For this particular $\vec{u}, \vec{v}$ we have $\vec{w} \in \mathrm{Span}(\vec{u}, \vec{v})$ if and only if $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent. Explain.

Theorem

If a set contains more vectors than there are entries (that is, rows) in the vectors, then it is linearly dependent.

Proof.

$\square$

Theorem

If $\vec{0} \in S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ then S is linearly dependent.

Proof.

$\square$

To test whether vectors $\vec{v}_1, \vec{v}_2, \dots \vec{v}_k$ are linearly dependent, construct the matrix $A = [\vec{v}_1, \vec{v}_2, \dots \vec{v}_k]$ having them as columns. Perform row reduction on the matrix.

If every column contains a pivot, then the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$, and hence the vectors are linearly independent, since then the only solution to

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k = \vec{0}$$

is $x_1 = x_2 = \cdots = x_k = 0$.

On the other hand, if there is a column without a pivot, then there are infinitely many solutions to the equation $A\vec{x} = \vec{0}$ (since there is a free variable in the general solution to the equation): pick a non-zero solution: it will correspond to a non-trivial solution to

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k = \vec{0}$$

and hence the vectors are linearly dependent.