MTHSC 3110 Section 1.8 – Linear Transformations

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We are going to talk about special kinds of functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ (note: n comes before m here!)
First: what is a function?

Definition

We write $f : A \longrightarrow B$ to denote that f is a function from the set A to the set B. What this means is that f is a rule, which assigns, for each element $a \in A$, a unique element $b \in B$ so that $b = f(a)$. We refer to A as the domain of f, and to B as the co-domain. The set

$$\{f(a) : a \in A\}$$

of values taken by the function is called the image or range of f.

We will restrict our attention to functions which interact nicely with vector addition and scalar multiplication.

Definition

A *linear transformation* from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ which satisfies the following two properties: whenever $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, then

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
2. $T(c\vec{u}) = cT(\vec{u})$.

Note

If A is a $m \times n$ matrix, then we can define $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\vec{x}) = A\vec{x}$.

We have already proved that
$T(\vec{x} + \vec{y}) = A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = T(\vec{x}) + T(\vec{y})$ and
$T(\alpha\vec{x}) = A(\alpha\vec{x}) = \alpha A\vec{x} = \alpha T(\vec{x})$ for any $\alpha \in \mathbb{R}$.

Example

Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 5 & -1 \end{pmatrix}$$

Define $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$.

1. $T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) =$
2. Find all $\vec{x} \in \mathbb{R}^2$ so that $T(\vec{x}) = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$.

Example

Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by $T(\vec{x}) = A\vec{x}$ is called a projection:

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

Example

Consider the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{x}) = A\vec{x}$ where $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. This is an example of a shear transformation. Draw the image of the box $[0,1] \times [0,1]$ under T to see why.

Note

If T is a linear transformation, then

1. $T(\vec{0} = \vec{0}$.
2. $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$.
3. $T\left(\sum_{i=1}^{p} c_i \vec{v}_i\right) = \sum_{i=1}^{p} c_i T(\vec{v}_i)$

Example

Define a map $T_r : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $T\vec{x} = r\vec{x}$. If $0 < r < 1$, the map is called a *contraction*. If $r > 1$ it is called a dilation. Show that T_r is a linear transformation.